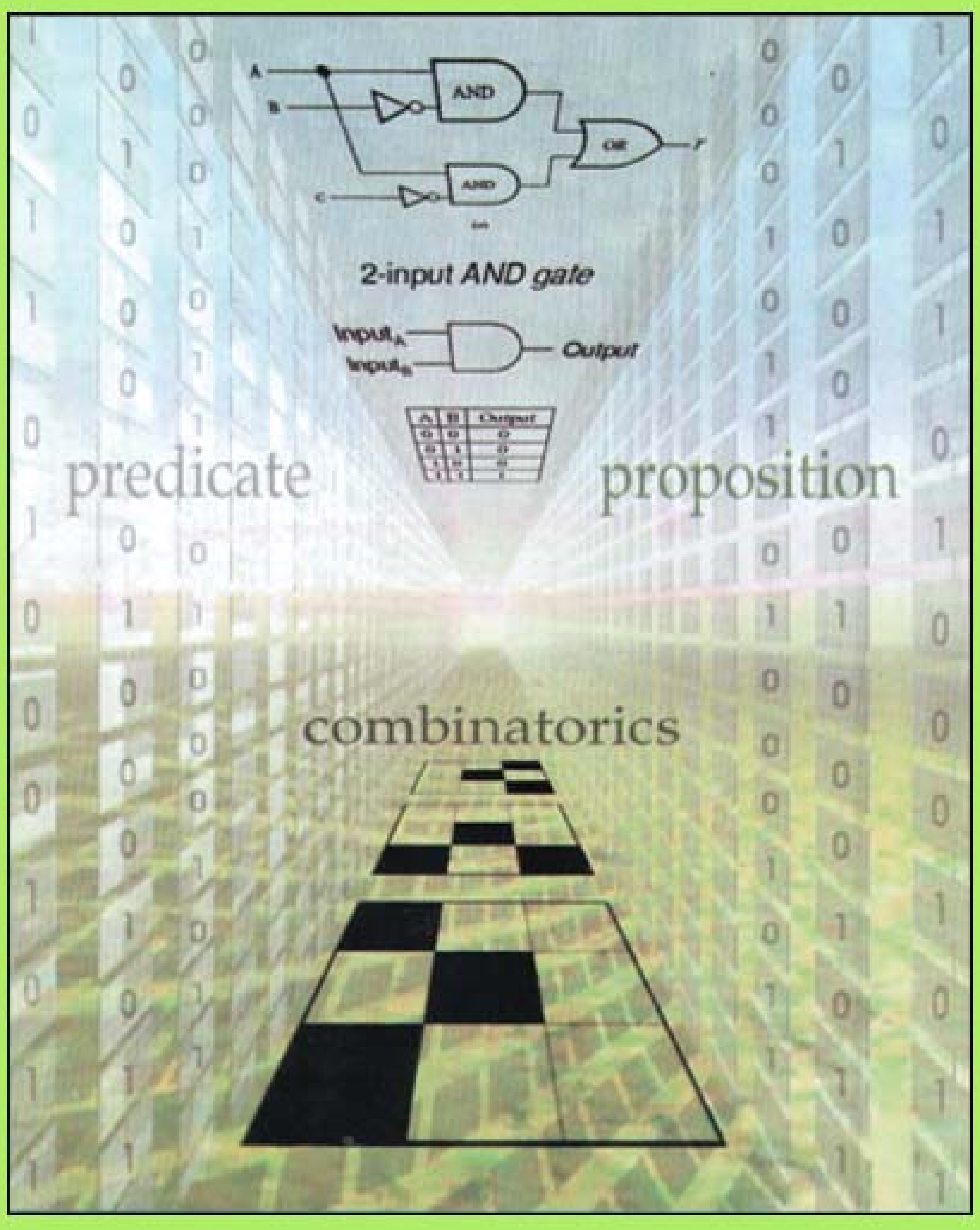
Indira Gandhi National Open University
School of Computer and Information Sciences

MCS-013

DISCRETE MATHEMATICS



Basic Combinatorics

2

Indira Gandhi
National Open University
School of Computer and
Information Sciences

MCS-013
DISCRETE MATHEMATICS

Block

2

BASIC COMBINATORICS

Programme / Course Design Committee

Prof. Sanjeev K. Aggarwal, IIT, Kanpur
Prof. M. Balakrishnan, IIT , Delhi
Prof Harish Karnick, IIT, Kanpur
Prof. C. Pandurangan, IIT, Madras
Dr. Om Vikas, Sr. Director, MIT
Prof P. S. Grover, Sr. Consultant, SOCIS, IGNOU

Faculty of School of Computer and Information Sciences
Shri Shashi Bhushan
Shri Akshay Kumar
Prof Manohar Lal
Shri V.V. Subrahmanyam
Shri P.Venkata Suresh

Block Preparation Team

Prof. Parvin Sinclair (Editor)
SOS, IGNOU

Shri M.P. Mishra
SOCIS, IGNOU

Partially based on Block-2, MTE-13.

Course Coordinator : Shri M.P.Mishra

Block Production

Shri H.K. Som, SOCIS

May, 2004

ISBN – 81- 266-1232-0

Further information on the Indira Gandhi National Open University courses may be obtained from the University's office at Maidan Garhi, New Delhi-110068.

Printed and published on behalf on the Indira Gandhi National Open University, New Delhi by the Director, SOCIS.

BLOCK 2 BASIC COMBINATORICS

Have you ever thought about how you can decide whether a given element belongs to a collection or not? Or, how a communication engineer can find the total number of distinct ways in which a fixed number of dots and dashes can be used for telegraphic communication? Or, how we can count the number of primes less than or equal to a given number? Problems such as these are what we discuss in this block. The focus of this block is a variety of techniques used for counting, that is, combinatorial techniques. We use them to study the problem of determining the size, and in some cases also the structure, of various sets that arise in such diverse applications as games theory, probability, and algorithm analysis.

This block consists of four units. The first unit deals with sets, relations and functions. In this we discuss the basic concepts moreover we also expose you to different representations of sets, including the Venn diagram invented by the English Priest and logician John Venn (1834-1923). Then we consider operations on sets, different types of relations, functions and operations and functions.

In second unit we deal with permutations and combinations, the binomial and multinomial theorems, and combinatorial probability. In this context, you might find it interesting to note that the notion of permutation can be found in the Hebrew work "Sefer Yetzirah" (i.e., The Book of Creation). This is a manuscript written by a mystic some time between 200 and 600 A.D. Also, the 'binomial theorem', which everybody is so familiar with, first appeared in the work of Euclid (300 B.C.) What is of further historical interest is that Blaise Pascal (1623-1662), published in the 1650s a treatise dealing with the relationships among binomial coefficients, combinations, and polynomials. These results were used by Jakob Bernoulli (1645-1705) to prove the general form of the binomial theorem.

The third unit of this block deals with the pigeonhole principle and the principles of inclusion and exclusion. The latter principle has an interesting history, being found in different manuscripts under such names as the "Sieve Method" or the "Principle of Cross Classification". A set-theoretic version of this principle, which concerned itself with set unions and intersection, is found in "Doctrine of Chances" (1718), a text on probability theory by Abraham De Moivre (1667-1754). Somewhat earlier, in 1713, Pierre Remond de Montmort (1678-1719) used the idea behind the principle in his solution of the problem of derangements.

On the other hand, the pigeonhole principle has no clear-cut mathematical origin. This is also known as the Dirichlet-drawer principle, after the famous German mathematician Dirichlet (1805-1859).

In the fourth, and last, unit of this block, we discuss partitions of natural numbers. We consider efficient techniques for counting the number of ways of distributing a finite number of objects into a finite number of containers, usually called boxes. It was Leonard Euler (1707-1783) who advanced the study of partitions of integers in his 1740 volume opus, "Introduction in Analysin Infinitorum".

Before we end, a note of advice! If you really want to get to grips with the content of this block, you must attempt the Miscellaneous Exercises given at the end of the block. Doing this, will help you understand the underlying reasoning better, and hence appreciate the theory of combinatorics.

NOTATIONS AND SYMBOLS

$\cap$	Intersection of two Sets
$\cup$	Union of two Sets
$\subseteq$	Subsets
$\supseteq$	Contains
~	Set difference
$\forall$	For all
$\exists$	There exists
ϕ	Empty Set
N	Set of natural numbers
R	Set of real numbers
Z	Set of integers
n!	n(n −1)...2.1
P(n,r)	$\frac{n!}{(n-r)!}$
C(n,r)	$\frac{n!}{(n-r)!r!}$
$P(n;r_1,r_2,\ldots r_k)$	$\frac{n!}{r_1!r_2!\ldots_k!}$
n(A),|A|	The cardinality of the set A
P(X)	The power set of the set X
P(A)	The probability of the event A
P_n	The number of partitions of the natural number n
P_n^k	The number of partitions of n with exactly k parts
$P_n(k)$	The number of partitions of n with no part larger than k
S_n^m $(n \geq m)$	The Stirling number of the second kind
$[x]_n$	x(x-1)(x-2)….(x-n+1), i.e., falling factorial
D_n	The number of derangements of n objects.

UNIT 1 SETS, RELATIONS AND FUNCTIONS

Structure **Page No.**

1.0 INTRODUCTION

In common parlance, we find people using the words given in the title of this unit. Do they have the same meaning in mathematics? You'll find this out by studying this unit. You will also see how basic the concept of 'set' and 'function' or to any area of mathematics and subjects depend on mathematics.

In this unit we will begin by introducing you to various kinds of sets. You will also study operations like, 'union' and 'intersection'. While doing so you will see in what way Venn diagrams are a useful tool for understanding and working with sets.

Next we will discuss what a relation is, and expose you to some important types of relations. You will come across while studying banking, engineering, information technology and computer science, of course mathematics. As you will see in your study of computer science, an extensive use of functions is made in problem-solving.

Finally, we lead you detailed discussion of functions. Over here we particularly focus on various points of functions and fundamental operations on functions.

1.1 OBJECTIVES

After studying this unit, you should be able to:

- explain what a set , a relation or a function is
- give examples and non-examples of sets, relations and functions
- perform different operations on sets
- establish relationships between operations on sets and those on statements in logic
- use Venn diagrams
- explain the difference between a relation and a function.
- describe different types of relations and functions.
- define and perform the four basic operations on functions

1.2 INTRODUCING SETS

In our daily life we encounter collections, like the collection of coins of various countries, a collection of good students in a class, a collection of faculty members of

IGNOU, etc. In the first of these examples, it is easy for anybody anywhere to tell whether an object belongs to this collection or not. If we take the collection of coins of a country, then a coin will be in the collection if it is a coin of that country, not otherwise. The criterion for being a member of the collection is objective and clear. However, if we take the collection of all good students, it is very difficult to say whether a person belongs to this collection or not because the characteristic *good* is not very clearly defined. In this case the collection is not 'well-defined', while the previous collection is 'well-defined'. Similarly, the collection of all the IGNOU students is well-defined.

Definition: A **set** is a well-defined collection. The objects belonging to a set are called **elements** or **members** of that set.

We write the elements of a set within curly brackets. For instance, consider the set A of stationary items used by Nazia. We write this as

A = {pen, pencil, eraser, sharpener, paper}

Another example is the set

B = {Lucknow, Patna, Bhopal, Itanagar, Shillong}

of the capitals of 5 states of India.

Note that A and B are well-defined collections. However, the collection of short people is not well-defined, and therefore, it is not a set.

Also note that **the elements of a set don't have to appear 'similar'**. For example, **{pen,Lucknow,4} is a set** consisting of 3 clearly defined elements.

As you have seen, we usually, denote sets by capital letters of the English alphabet. We usually denote the elements by small letters a,b,x,y …. If x is an element of a set A, we write this as $\mathbf{x \in A}$ (read as 'x belongs to A'). If x is not an element of A, we write this as $x \notin \mathbf{A}$ (read as 'x does not belong to A').

There are three ways of representing sets: 'Set-builder form', 'Tabular form' and the pictorial representation through Venn diagrams.

In the **'Set-builder form',** or **'property method'** of representation of sets, we write between brackets { } a variable x, which stands for each of the elements of the set which have the properties p(x), and separate x and p(x) by a symbol ':' or '|' (read as 'such that'). So the set looks like {x: p(x)} or {x | p(x)}.

For instance, the set {x | x is a white flower} is the set of all white flowers, or {x: x is a natural number and $2<x<11$} is the set of natural numbers lying between 2 and 11.

In **'Tabular form'**, or the **'listing method'**, the elements of a set are listed one by one within the brackets { }, each separated from the other by a comma, as in the examples A and B given above.

The accepted convention for writing a set by the listing method is that elements will not be repeated. For example, in the set A= {4,2,8,2,6}, 2 is repeated, which is not necessary.So we will write A ={4,2,8,6}.

We shall introduce you to Venn diagrams a little later. For now, let us consider a few more sets.

Definition: A set with no element is called the **empty(**or **null**, or **void) set**, and is denoted by ϕ or{}.

For example A = {x:x is an integer between 13 and 17 which is divisible by 6}, has no element, i.e., A is the **empty set.**

Definition: A set having a finite number of elements is called a **finite set**.

For example, $\{1,2,4,6\}$ is a finite set because it has four elements, ϕ, the null set, is also a finite set because it has zero number of elements; the set of stars in the sky is also a finite set.

Definition: A set having infinitely many elements is called an **infinite set**.

For example, the set **N** of natural numbers is infinite. Similarly, **Q , Z, R** and **C**, the set of rational numbers and complex numbers, respectively, are infinite set.

B= The set of all strengthness in a given plane.

Now try the following exercises.

E1) How would you represent the set of all students who have offered the IGNOU course?

E2) Explain, with reason, whether or not

i) the collection of all good teachers is a set

ii) the set of points on a line is finite.

E3) Represent the set of all integers by the listing method.

When we deal with several sets, we need to understand the nature of the elements of those sets, whether the elements of two given sets have some elements in common or not, and so on. These questions involve concepts, which we now define.

Definition: A set A is said to be a **subset** of a set B if each element of A is also an element of B. In this case B is called a **superset** of A. If A is a subset of B, we represent this by $\mathbf{A \subseteq B}$.

As a statement in logic we represent this situation as,

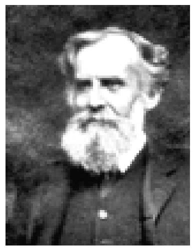
$A \subseteq B \Leftrightarrow [x \in A \Rightarrow x \in B]$

'B contains A' or 'B is a superset of A' is represented by $B \supseteq$ **A.**
If A is not a subset of B, we represent this by $\mathbf{A \not\subseteq B}$.

For example, if A $=\{4,5,6\}$ and B $=\{4,5,7,8,6\}$, then $A \subseteq B$. But if C = $\{3,4\}$ then $C \not\subseteq B$.

Remark: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition: Two sets A and B are **equal** if every element of A belongs to B and every element of B belongs to A. We represent this by **A=B**.



(1834 -1923)
Fig 1: John Venn

For example, if A= $\{1,2,3\}$, B $=\{2,3,1\}$, then $A \subseteq B$ and $B \subseteq A$, so that A = B.

Definition: A set A is said to be a **proper subset** of a set B if A is a subset of B and A and B are not equal. We represent this by $\mathbf{A \subset B}$.

For example, if A $=\{4,5,6\}$ and B = $\{4,5,7,8,6\}$, then $A \subset B$; and if A={Java, C, C++, Cobol} and B={Java, C++}, then $A \supset B$.

Note: A set can have many subsets and many supersets. For example A=$\{1,2,3,4,5\}$, B= $\{2,3,4,5,6,7\}$, and C= $\{2,3\}$, then for C, A and B can be used as supersets.

Similarly, if X= {Ram, Rani, Sita, Gita}, Y= {Rani}, and Z= {Sita}, then Y and Z both are subsets of X.

Definition: The **power set** of a set A is the set of all the subsets of A, and is denoted by **P(A)**.

Mathematically, $P(A) = \{ x : x \subseteq A\}$.

Note that $\phi \in P(A)$ and $A \in P(A)$ for all sets A.

For example, if $A=\{1\}$, then $P(A)= \{\phi, \{1\}\}$ and

if $A= \{1,2\}$, then $P(A)=\{\phi, \{1\}, \{2\}, \{1,2\}\}$

Similarly, if $A = \{1,2,3\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Definition: Any set which is a **superset** of all the sets **under consideration** is known as the **universal set**. This is usually denoted by Ω**, S** or **U.**

For example, if $A = \{1,2,3\}$, $B = \{3,4,6,9\}$ and $C = \{0,1\}$, then we can take $U = \{0,1,2,3,4,5,6,7,8,9\}$ or U= **N**, or U=**Z** as the universal set.

Note that the universal set can be chosen arbitrarily for a given problem. But once chosen, it is fixed for the discussion of that problem.

Theorem 1: If A is a set with n elements, then $|P(A)| = 2^n$.

Proof: We shall prove this by mathematical induction.
For this, we first check if it is true for n= 1.Then assuming that it is true for n = m, we prove it for the case n = m +1. It will, then, follow that the result will be true $\forall$ n $\in$**N**.

Step I: If A = 1, then P(A) = 2=2^1.

Step II: Assume that the theorem holds for all sets A of cardinality k, i.e. if $|A| = k$, then A has 2^k subsets.

Step III: Now consider any set $A = \{x_1, x_2, x_3,\ldots,x_k, x_{k+1}\}$, with k+1 elements. Consider its subset $B= \{x_1, x_2, x_3,\ldots,x_k \}$. Now B has 2^k subsets, each being a subset of A. Now, take any such subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ of B. Then $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, x_{k+1}\}$ is a subset of A that is not a subset of B. So, for each of the 2^k subsets of B, we attach x_{k+1} to it to get 2^k more subsets of A.

You can see that this covers all the subsets of A.

So the number of subsets of A=$2^k + 2^k = 2.2^k = 2^{k+1}$.
Hence the theorem.

Now try these exercises.

E4) Give two proper subsets and two supersets of the set of vowels of the English alphabet.

E5) Find the power set of the set A ={a,e,i,o,u}.

E6) For which set A, is P(A) = 1 ?

E7) If $A \subseteq B$, is $P(A) \subseteq P(B)$? Why?

E8) $P(A) = P(B) \Rightarrow A = B$. True or false? Why?

Let us conclude this section with the **pictorial** representation of sets. You know that the pictorial representation of any object helps in understanding the object. This is why a pictorial representation of sets, known as a **Venn diagram,** helps in understanding and dealing with sets.

The English priest and logician John Venn invented the Venn diagram. Through Venn diagrams we can easily visualize the abstract concept of a set and operations on sets. In this diagram, the universal set is usually represented by a rectangle and its subsets are shown as circles or other closed geometrical figures inside this rectangle.

For example, A = {Lucknow, Patna, Bhopal, Itanagar, Shillong} can be represented using a Venn diagram as in Fig. 2. Here U could be any superset of A.

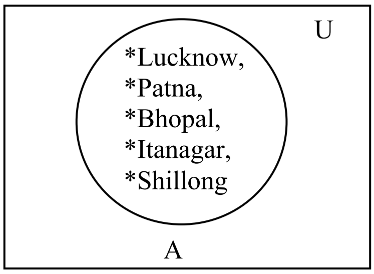


Fig. 2: A Venn diagram

Now that you are familiar with basic definitions related to sets, let us discuss some basic operations that can be performed on sets. This is when we shall appeal to Venn diagrams very often, as you will see.

1.3 OPERATIONS ON SETS

Let us now study sets obtained by applying operations on sets. We will cover four operations here, namely, union of sets, intersection of sets, complement of sets and symmetric difference. While studying them you will see how useful a Venn diagram can be for proving results related to these operations. In this section we will also look at some rules that are common to operations on sets and operations on statements, which you studied in Block 1.

1.3.1 Basic Operations

In this sub-section we shall define each of the operations one by one.

Definition: The **union** of two sets A and B is the set of all those elements which are either in A or in B or in both A and B. This set is denoted by $\mathbf{A \cup B}$, and read as ‘A union B’.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

For example, if A = {x: x is a stamp} and B = {4,5}, then

$A \cup B$= {x:x is a stamp or a natural number lying between 3 and 6}.

And A={Ram, Mohan, Ravi} and B={Ravi, Rita, Neetu},
then $A \cup B$={Ram, Mohan, Ravi, Rita, Neetu}.

If $A \subseteq B$, then $A \cup B = B$, and vice versa. This can be shown immediately using a Venn diagram, as in Fig.3.(a), where A is shown as the square contained in the circle representing B. In Fig.3(b), $A \cup B$ is shown when A and B have some elements in common, and in Fig.3(c), we depict $A \cup B$ when A and B have no element in common.

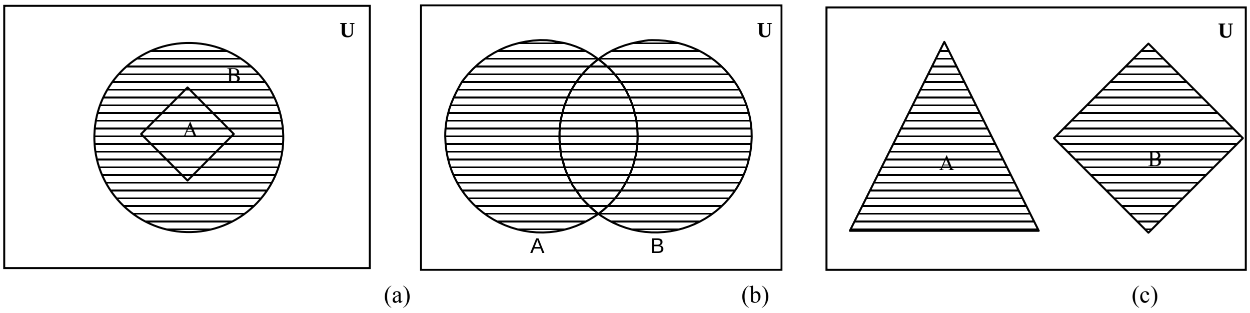


Fig. 3: Venn diagram for union

Definition: The **intersection** of sets A and B is the set of all the elements which are common to both A and B. This set is denoted by **A $\cap$ B**, and read as 'A intersection B'.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$;
For example $A = \{1,2,3\}$ and $B = \{2,1,5,6\}$, then $A \cap B = \{1,2\}$.
Again if $A = \{1\}$ and $B = \{5\}$ then $A \cap B = \{\ \}$ or ϕ.

Remark: For any two sets A and B, $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$.

What is $A \cap B$ if $A \subseteq B$? Do you agree that it is A? Let us use a Venn diagram to check this (see Fig.4(a)). If A and B have some elements in common, then the Venn diagram for $A \cap B$ looks like Fig 4.(b), and if A and B have no element in common, then the Venn diagram will be as in Fig.4(c).

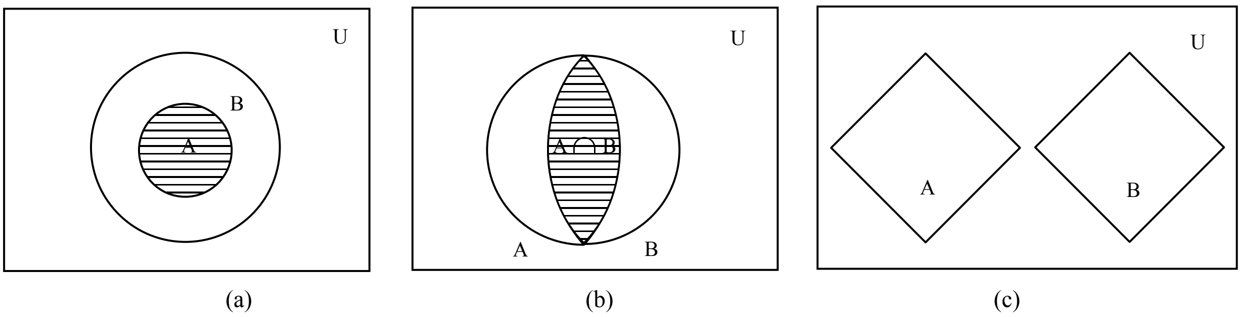


Fig. 4: Venn diagram for intersection of sets

Definition: The **difference of two sets** A and B is the set of all those elements of A which are not elements of B. Sometimes, we call this set the **relative component** of B in A. It is denoted by **A~B** or **A\B**, and is read as 'A complement B'.

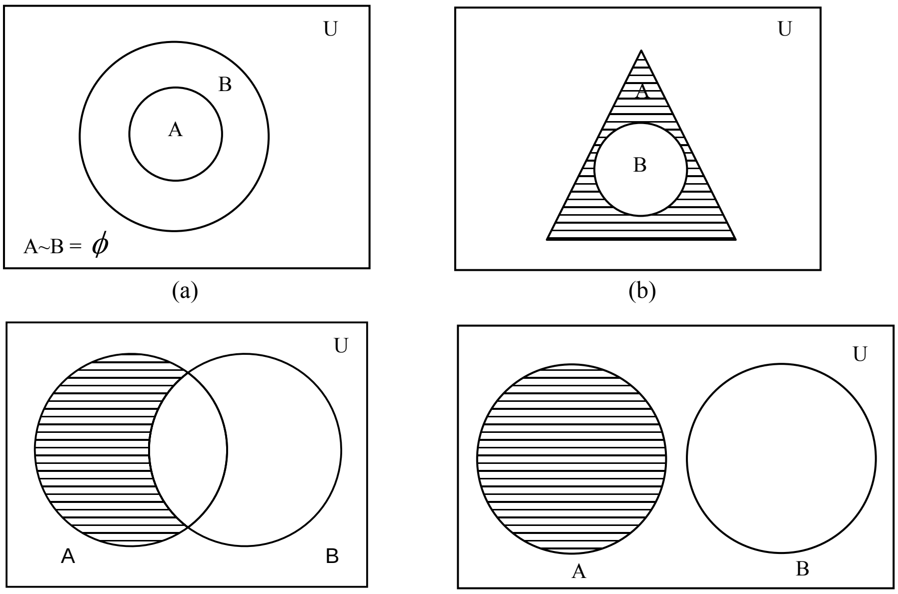
Symbolically, $A \sim B = \{x : x \in A \text{ and } x \notin B\}$ and
$B \sim A = \{x : x \in B \text{ and } x \notin A\}$

For example, if $A = \{4,5,6,7,8,9\}$ and $B = \{3,5,2,7\}$, then $A \sim B = \{4,6,8,9\}$ and $B \sim A = \{3,2\}$. From this example it is clear that $A \sim B \neq B \sim A$. In fact, this is usually the case. So, **the difference of sets is not a commutative operation**.

In Fig.5(a), $A \subseteq B$, so that $A \sim B = \phi$.

In Fig.5(b) we show A~B when $A \supseteq B$, and in Fig.5(c) we show A~B when neither $A \subseteq B$ nor $B \subseteq A$.

In Fig. 5(d), we show A~B when A and B are disjoint.



(c) (d)

Fig. 5: A ~ B in different situation is the shaded portion.

There is one particular 'difference' that shows up very often, which we now define.

Definition: The **complement of a set** A, is the set U \ A, and is denoted by A′ or A^c. For example, U={Physics, Chemistry, Mathematics} and A={Mathematics}, then the complement of A is A′= {Physics, Chemistry}.

The Venn diagram showing the complement of Á is the set of those elements of the universal set U which are outside A (see Fig.6).

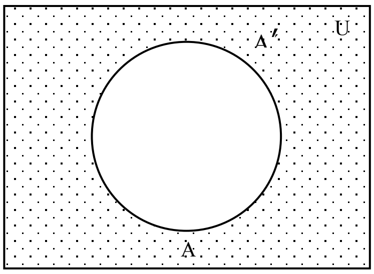


Fig. 6: Venn diagram for A′.

Definition: The **symmetric difference** of two sets A and B is the set of all those elements which are in A or in B, but not in both. It is denoted by A Δ B.

i.e., A Δ B =(A~B) $\cup$ (B~A).

Note that A Δ B = B Δ A, i.e. the symmetric difference is commutative.

For example A= {1,2,3,4,5} and B= {3,5,6,7}, then A~ B= {1,2,4},and B~A = {6,7}
$\therefore$ A Δ B={A~B) $\cup$ (B~A) = {1,2,4,6,7}

Now you may try these exercises.

E 9) Make a Venn diagram for A Δ B for each of the situations i) A $\subseteq$ B, ii) A $\not\subseteq$ B, iii) B $\not\subseteq$ A and A $\cap$ B $\neq \phi$; iv) A $\cap$ B= ϕ .

E10) Let A= {Math, Physics, Science}, B={Computer,Math,Chemistry}, C={Math}. Find A $\cup$ (B $\cap$ C).

E11) If A = {1,2,3,4,5,6}, B={4,5,6,7,8,9},find i) A~ B, ii) B~ A, iii) A Δ B.

E12) For which sets A and B would A~B = B~A ?

E13) Write a program in C to perform E 10.

E14) Under what conditions can $A \cap B = A \cup B$?

While discussing these operations, you may be wondering that they seem to satisfy properties very similar to those of propositional logic covered in Block 1 of this course.You are right! Let us look at this aspects now.

1.3.2 Properties Common to Logic and Sets

Before looking into the properties we shall first present a very useful principle to you. This will allow you to see how one property can be proved in several situations simultaneously.

Duality Principle: The 'duality principle' is very convenient for dealing with theorems about sets. Basically if any theorem is given to you, by applying the duality principle you can get another theorem (the dual of the previous one). In any statement involving the union and intersection of sets, we can get from one of the statements to the other by interchanging $\cap$ with $\cup$ and ϕ with U.

For example, the dual of $A \cup (B \cap C)$ is $A \cap (B \cup C)$ and the dual of $U \cup \phi = U$ is $U \cap \phi = \phi$. So, for example what is true for $A \cup (B \cap C)$ will be true for $A \cap (B \cup C)$ too. This is why if the first property in each of the pairs below is proved the second one follows immediately.

For any universal set U and subsets A, B and C of U, **the following properties hold**.

i) Associative properties:
Union: $A \cup (B \cup C) = (A \cup B) \cup C$
Intersection: $A \cap (B \cap C) = (A \cap B) \cap C$

ii) Commutative properties:
Union: $A \cup B = B \cup A$
Intersection: $A \cap B = B \cap A$.

iii) Identity:
Union: $A \cup \phi = A$
Intersection: $A \cap U = A$.

iv) Complement:
Union: $A \cup A' = U$
Intersection: $A \cap A' = \phi$

v) Distributive properties:
Union: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Intersection: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Fig. 7: Augustus De Morgan (1806–1871)

De Morgan's Laws:

For any two sets A and B the following laws known as De Morgan's laws, hold

1. $(A \cup B)' = A' \cap B'$, and
2. $(A \cap B)' = A' \cup B'$

De Morgan's laws can also be expressed as

1. $A \sim (B \cup C) = (A \sim B) \cap (A \sim C)$
2. $A \sim (B \cap C) = (A \sim B) \cup (A \sim C)$

Each of the properties above corresponds to a related property for mathematical statements in logic (which we have covered in Unit 2 and Unit 3 of Block 1 of this course).

Now try these exercises.

E15) Find the dual of

i) $A \cap (B \cap C) = (A \cap B) \cap C$, and ii) $(A \cup B) \cap (A \cup C)$.

E16) Draw a Venn diagram to represent $A \cup (B \cap C)$.

E17) Check whether $(A \cup B) \cap C = A \cup (B \cap C)$ or not using a Venn diagram.

Let us now focus on subsets of a particular kind of product of sets.

1.4 RELATIONS

Sometimes we need to establish relations between two or more sets. For example, a software development company has a set of specialists in different technology domains, or a company gets some projects to develop. Here the company needs to establish a relation between professionals and the project in which they will participate. To solve this type of problem the following concepts are required.

1.4.1 Cartesian product

Very often we deal with several sets at a time, and we need to study their combined action. For instance, combinations of a set of teachers and a set of students. In such a situation we can take a product of these sets to handle them simultaneously. To understand this product let us first consider the following definitions.

iff is short for 'if and only if'.

Definition: An **ordered pair**, usually denoted by (x,y), is a pair of elements x and y of some sets. This is ordered in the sense that $(x,y) \neq (y,x)$ whenever $x \neq y$, that is , the order of placing of the element in the pair matters.

Any two ordered pairs (x,y) and (a,b) are equal iff x = a and y = b.

For example if, A= {a,b,c}and B= {x,y,z},then

$A \times B = \{a,b,c\} \times \{x,y,z\} = \{(a,x), (a,y),(a,z),(b,x),(b,y),(b,z),(c,x),(c,y), (c,z)\}$,and

$B \times A = \{x,y,z\} \times \{a,b,c\} = \{(x,a),(x,b),(x,c),(y,a),(y,b),(y,c),(z,a),(z,b),(z,c)\}$.

Now let us think about how $B \times A$ can be represented geometrically? For instance what is the geometric view of $\{2\} \times \mathbf{R}$? This is the line x =2 given in Fig.8(a).

Now, after seeing geometric representation of $\{2\} \times \mathbf{R}$, can you tell what $\{1,3\}\times\{2,3\} = \{(1,2),(3,2),(1,3),(3,3)\}$ looks like? You will get four points in the first quadrant, as shown in Fig.8(b).

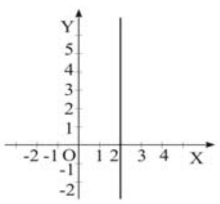


Fig. 8(a): {2} × R, i.e., x = 2.

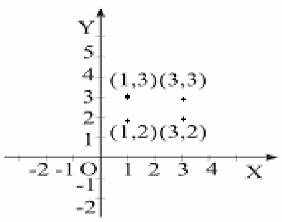


Fig.8(b): {1,3}×{2,3}

Now, you know that the multiplication of numbers is commutative. Is the Cartesian product of sets also commutative? For instance, is $\{1\}\times\{2\}=\{2\}\times\{1\}$? No, because $(1,2) \neq (2,1)$. So, **A×B≠ B×A usually**.

We can extend the definition of A × B to define the Cartesian product of n sets A_1, A_2, ….,A_n as follows.

$A_1 \times A_2 \times A_3 \times \ldots \times, A_n = \{(x_1,x_2,x_3,\ldots,x_n)\}: x_1 \in A_1, \wedge\ x_2 \in A_2 \wedge x_3 \in A_3 \wedge \ldots, \wedge\ x_n \in A_n\}$.

The element $(x_1,x_2,\ldots,x_n)$ is called an **n-tuple.** For instance, the 3–tuple $(1,1,3) \in \{1\} \times \{1,2\} \times \{2,3\}$.

Now you may try some exercises.

E18) If X= {a,b,c} and Y={1,2,3},find

i) $X \times X$, ii) $X \times Y$, and iii) $X \times \phi$.

E19) Under what conditions on A and B is A×B = B×A ?

E20) Give the geometric representation of $\mathbf{R} \times \{2\}$.

With what you studied in this sub-section, you now have the background to discuss relations.

1.4.2 Relations and their Types

We often speak of relations which hold between two or more objects, e.g., discrete mathematics is one of the courses in the IGNOU MCA Ist semester, Nehru wrote Freedom of India, Chennai is the capital of Tamil Nadu. These are the relations in everyday situations. In these examples some sort of connections between pairs of objects are shown, and hence they express a relation between the pairs of objects.

Definition: A **relation between two sets A and B** is a subset of A × B. Any subset of A × A is **a relation on the set A**.

For instance, if A ={1,2,3} and B = {p,q}, then the subset {(1,p),(2,q),(2,p)} is a relation on A × B. And {(1,1), (2,3)} is a relation on A.

Also, $R= \{(x,y) \in \mathbf{N} \times \mathbf{N} : x > y\}$ is a relation on **N,** the set of natural numbers, since $R \subseteq \mathbf{N} \times \mathbf{N}$.

If $R \subseteq A \times B$,. we write $_{x}\mathbf{R}_{y}$ if and only if $(x,y) \in R$($_{x}R_{y}$ is read as ‘x is related to y’).

Theorem 2: The total number of distinct relations between a finite set A and a finite set B is 2^{mn}, where m and n are the number of elements in A and B, respectively.

For example, $R_1 = \mathbf{N} \times L$, where L is set of straight lines, in this relation we can give different ordering of the straight lines.
If the relation $R_2 = \{1,2,3\} \times \{l_1,l_2\}$, then line l_1 and l_2 can get three different ordering.

Proof: The number of elements of A × B is mn. Therefore, the number of elements of the power set of A× B is 2^{mn} (See Theorem 1). Thus, A × B has 2^{mn} different subsets. Now every subset of A× B is a relation from A to B, by definition. Hence the number of different relations from A to B is 2^{mn}.

As you have seen, any and every subset of $A \times A$ is a relation on A. However, some relations have special properties. Let us consider these types one by one.

1.4.3 Properties of Relations

Reflexive Relations: A relation R on a set A is called a **reflexive relation** if $(a,a) \in R \; \forall \; \mathbf{a} \in \mathbf{A}$.

In other words, R is reflexive if every element in A is related to itself.
Thus, R is **not reflexive** if there is at least one element $a \in A$ such that $(a,a) \notin R$.

For example, if A= {1,2,3,4}, then the relation R_1= {(1,1), (2,2), (3,3), (4,4)} in A is reflexive because for $x \in A,(x,x) \in R_1$. However, R_2= {(1,1), (2,1), (4,4)} is not reflexive since $2 \in A$, but $(2,2) \notin R_2$.

Symmetric Relations: A relation R on a set A is called a **symmetric relation** if $(a,b) \in R \Rightarrow (b,a) \in R$.Thus, R is symmetric if bRa holds whenever aRb holds.

A relation R in a set A is **not symmetric** if there exist two distinct elements a, $b \in A$, such that aRb, but not bRa.

For example, if L is the set of all straight lines in a plane, then the relation R in L, defined by 'x is parallel to y', is symmetric, since if a straight line a is parallel to a straight line b, then b is also parallel to a. Thus, $(a,b) \in R \Rightarrow (b,a) \in R$.

However, if R is the relation on **N** defined by 'xRy iff x–y>0', then R is not symmetric, since, 4−2>0 but $2-4 \not> 0$.Thus, $(4,2) \in R$ but $(2,4) \notin R$.

Transitive Relations: A relation R on a set A is called a **transitive relation** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for $a,b,c \in A$.
Thus, $[(a,b), \in R,(b,c) \in R \Rightarrow (a,c) \in R]$, $\forall \; a,b,c \in A \Rightarrow$ R is transitive.

A relation R in a set A is **not transitive** if there exist elements $a,b,c \in A$, not necessarily distinct, such that $(a,b) \in R$, $(b,c) \in R$ but $(a,c) \notin R$.

For example, if L is the set of all straight lines in a plane and R is the relation on L defined by ' x is parallel to y' then if a is parallel to b and b is parallel to c, then a is parallel to c. Hence R is transitive. However, the relation 'xSy'on L defined by 'iff x intersects y' is not transitive.

Also, the relation R on A, the set of all Indians, defined by 'xRy iff x loves y', is not a transitive relation.

Equivalence Relations: A relation R on a set A is called an **equivalence relation** if and only if

(i) R is reflexive, i.e., for all $a \in R$, $(a, a) \in R$,

(ii) R is symmetric, i.e., $(a, b) \in R \Rightarrow (b, a) \in R$, for all a, $b \in A$, and

(iii) R is transitive, i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, for all a, b, $c \in A$.

One of the most trivial examples of an equivalence relation is that of **'equality'.**
For any elements a,b,c in a set A,

(i) a = a, i.e., reflexivity

(ii) $a = b \Rightarrow b = a$, i.e., symmetricity

(iii) a = b and b = c $\Rightarrow$ a = c , i.e., transitivity.

Now let us see if 'xRy iff ' '$x \le y$' gives an equivalence relation on **R**.

(i) $x \le x$, i.e., $(x,x) \in \mathbf{R}$, i.e., R is reflexive.

(ii) However, $2 \le 3$ but $3 \nleq 2$. So, R is not symmetric.

Thus, R is not an equivalence relation.

Now you may try these exercises.

E 21) Let A be the set of all people on Earth. A relation R is defined on the set A by 'aRb if and only if a loves b' for $a,b \in A$.

Examine if R is i) reflexive, ii) symmetric, iii) transitive.

Now we shall study a particular kind of relation, which is very useful in mathematics, as well as in computer science, as you will soon see.

1.5 FUNCTIONS

A function is a special kind of relation. If we take the example of the set A of students of IGNOU, and the set B of their enrolment numbers. Now consider R = $\{(a,b) \in A \times B$ | b is enrollment number of a $\}$,this is a relation between A and B. It is a 'special' relation, 'special' because to each $a \in A \exists\ !\ b$ such that aRb. We call such a relation a function from A to B.

Let us define this term formally.

Definition: A **function from a non-empty set A to a non-empty set B** is a subset R of $A \times B$ such that for each $a \in A \exists$ a unique $b \in B$ such that $(a,b) \in R$. So, this relation satisfies the following two conditions:

(i) for each $a \in A$, there is some $b \in B$ such that $(a,b) \in R$

(ii) if $(a,b) \in R$ and $(a,b') \in R$ then $b = b'$.

We usually present functions as a rule associating elements of one set with another. So, let us present the definition again, with this view.

Definition: Let A and B be non-empty sets. A **function** (or a **mapping**) f from A to B is a rule that assigns to each element x in A **exactly one** element y in B. We write this as $f: A \rightarrow B$, read it as 'f is a function from A to B'.

Note that

(i) to each $a \in A$, f assigns an element of B; and

(ii) to each $a \in A$, f assigns only **one element** of B.

So, for example, suppose A =$\{1,2,3\}$, B= $\{1,4,9,11\}$ and f assigns to each member in A its square values. Then f is a function from A to B. But if A=$\{1,2,3,4\}$, B=$\{1,4,9,10\}$ and f is the same rule, then f is not a function from A to B since no member of B is assigned to the element 4 in A.

Note that the former example, $11 \in B$, but there is no element in A which is assigned to 11. This does not matter. It is not necessary that every element of B be related to some element of A.

Functions are not restricted to sets of numbers only. For instance, let A be the set of mothers and B be the set of human beings. Then the rule that assigns to every mother her eldest child is a **function**. But the rule that assigns to each mother her children is **not** a **function** because it does not relate a unique element of B to each element of A.

Now, given a function, we have certain sets and terms that are associated with it. Let us give them some names.

Definitions: Let f be a function from A to B. The set A is called the **domain** of the function f and B is called the **co-domain** of f. The set $\{f(x) | x \in A\}$ is called the **range** of f, and is also denoted by f(A).

Given an element $x \in A$, the unique element of B to which the function f associates, it is denoted by f (x) and is called the **f-image** (or **image**) of x or the value of the function f for x. We also say that f **maps** x to f(x). The element x is referred to as the **pre-image** of f (x).

For example, if A = {1,2,3,4}, B = {1,8,27,64,125}, and the rule f assigns to each member in A its cube, then f is a function from A to B. The domain of f is A, its codomain is B and its range is {1,8,27,64}.

Can you tell what will be the domain and codomain for rule f : $f(x) = \dfrac{x}{1-x}$?

You can see that 1–x = 0, if x = 1, in this case f(x) will be undefined.
Domain of f can be taken as R~{1}and codomain can be **R**.

Remark: Each element of A has a **unique image,** and each element of B need not appear as the image of an element in A. Further, more than one element of A can have the same image in B.

Let us look at some examples of functions, and non-functions now.

i) If b is be a fixed element of B, then f: A→B : f(x)= b $\forall$ x $\in$ A is called a **constant function**.

Note that if b=0, then f is called the **zero map**, and is denoted by **0**.

ii) f : A→ A : f(x) = x $\forall$ x $\in$ A is called the **identity function** , and is denoted by **I**.

iii) Consider A= {1,2,3,4}, B= {1,4,5} and the rule f which associates 1→1, 2→ 4, 3→5, 4→5. Then f is a function from A to B.

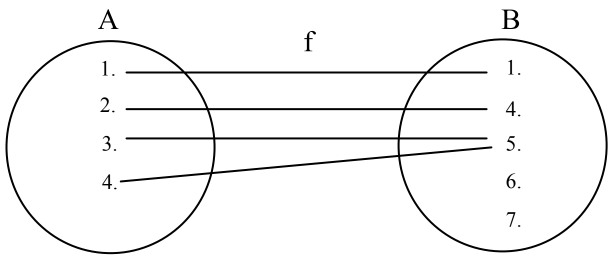


Fig.11: The rule f is a function

iv) The function f from **R** to **Z**, defined by the rule that f maps any real number x to the greatest integer less than or equal to x. is known as the **greatest integer** function or the **floor function**. We denote this function's action by f(x)=[x], where [x] is the greatest integer $\leq$x.

For example, if x = 0.6 then f(x) =[x]=0, if x= 2.3 then f(x)=[x]=2, and if x = –5, then [x]′= –5 .

v) Function f: **R**→**R**: f(x)=|x|is known as the modulus (or absolute value) function, where |x|is the absolute value of x.

For example, if x=10 then f(x)=|x|=10 and if x= –10, then f(x)=|x|=10.

vi) Now take, A={a,b,c} and B={1,2,3,4,5}. Consider the rule f which associates a→1, a→ 3, b→2, c→3. This is not a function from A to B because, elements 1 and 3 $\in$ B are assigned to the same element a $\in$A.

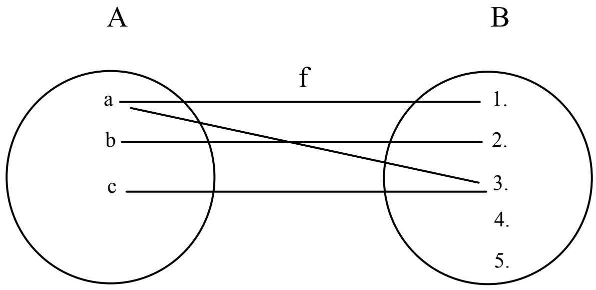


Fig.9: The rule f is not a function

vii) Consider A= {1,2,3}, B= {1,4,5,6,7} and the rule f which associates 1→1, 2→ 1, 2→4. Here f is not a function from A to B since no member of B is assigned to the element 3$\in$A.

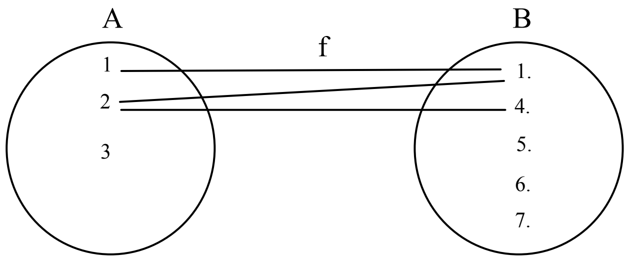


Fig.10: The rule is not a function

Now you may try these exercises.

E22) Let A= {a,b,c,d}, B= {1,2,3} and R= {(a,2), (b,1), (c,2), (d,1)}. Is R a function? Why.

E23) Every function is a relation. Is every relation a function? Why?

E24) Consider the following pseudocode.

1. read(n)
2. while n >1 do
3. begin
4. if n is even then n := n div 2
5. else n := 2n + 1;
6. end

Write a function of n that describes the operations performed.

E25) If A= {1,2,3,4}, B = {2,3,4,5,6,7} and the rule f assigned to each member in A is f(x)= x + 1, then find the domain and range of f.

Now let us discuss some types of functions.

1.5.1 Types of Functions

Here we shall look at different types of mappings.

'Surjective' comes from the French word 'sur', meaning 'on top of '.

Onto Mapping: A mapping f: A→ B is said to an **onto** (or **surjective**) **mapping** if f(A)= B, that is, the range and co-domain coincide. In this case we say that **f maps A onto B**.

For example, f: **Z**→ **Z** : f(x) = x+1, $x \in$ **Z**, then every element y in the co-domain **Z** has a pre-image y–1 in the domain **Z**. Therefore, f(**Z**) = **Z** ,and f is an onto mapping.

Injective Mapping: A mapping f: A→ B is said to be **injective** (or **one-one**) if the images of distinct elements of A under f are distinct, i.e., if $x_1 \neq x_2$ in A, then $f(x_1) \neq f(x_2)$ in B. This is briefly denoted by saying f is **1–1**.

For example f: **R**→ **R** be defined by f(x) = 2x+1, $x \in$ **R**, then for $x_1, x_2 \in$ **R** $(x_1 \neq x_2)$ we have $f(x_1) \neq f(x_2)$. So, f is **1–1**.

Bijective Mapping: A mapping f: A→ B is said to be **bijective** (or **one-one onto,**) if f is both injective and surjective, i.e., one-one as well as onto.

For example, f: **Z**→ **Z** : f(x) = x+2, $x \in$ **Z** is both injective and surjective. So, f is bijective.
There is a particular kind of bijective function that we use very often. Let us define this.

Definition: A bijective mapping f : A→ A is said to be a **permutation** on the set A. Let A = $\{a_1,a_2,\ldots,a_n\}$, and f be a bijection from A onto A that maps a_i to $f(a_i)$, then we write f as
$f = \begin{pmatrix} a_1 & a_2 \ldots a_n \\ f(a_1) & f(a_2)\ldots f(a_n) \end{pmatrix}$. So, the identity mapping I= $\begin{pmatrix} a_1 & a_2\ldots a_n \\ a_1 & a_2\ldots a_n \end{pmatrix}$.

Now, associated with a bijective function, we get another function very naturally, which we now define.

Definition: Let f: A→ B be a bijective mapping. Then the mapping g : B→ A which associates to each element $b \in$ B the unique element $a \in$ A, such that f(a) = b, is called the inverse mapping of the mapping f: A→ B. We denote this function g by f^{-1}.

Note that a function f is invertible iff f^{-1} exists iff f is bijective.
Hence, if f: A→B is a one-one onto mapping, then f^{-1} : B→A exists, and is also 1-to-1.
Note the inverse of the permutation f = $\begin{pmatrix} a_1 & a_2\ldots a_n \\ b_1 & b_2\ldots b_n \end{pmatrix}$ is the permutation
$\begin{pmatrix} b_1 & b_2\ldots b_n \\ a_1 & a_2\ldots a_n \end{pmatrix}$.

For example, A= **R**~{3} and B= **R**~(1}, and the function f: A→ B is defined by
$f(x)=\dfrac{x-2}{x-3}$.

We can see that f is a one–to–one function.
∴ f inverse exists.

To get $f^{-1}(x)$ the following steps are required;

1. Replace f(x) by y in the equation describing the function. You will get

   $y = \dfrac{x-2}{x-3}$.

2. Interchange x and y. In other words, replace every x by y, and vice versa. You will get $x = \dfrac{y-2}{y-3}$.

3. Solve for y.

4. Replace y by $f^{-1}(x)$.

By applying these steps we get $f^{-1}(x) = \dfrac{3x-2}{x-1}$.

Now try these exercises.

E26) Explain why $f : \mathbf{Z} \rightarrow \mathbf{Z}$: $f(x) = x^2$ is onto? Domain and range of f is **Z**.

E27) Which of the following kind of function would you use to provide photo identity numbers? Why?

i) Constant function, ii) one-to-one function, and iii) identity function.

E28) Find f inverse of rule $f : f(x) = x^3 - 3$.

Now we can see how different operations like addition, subtraction, multiplication and division can be applied on functions.

1.5.2 Operations on Functions

If given whose domains ranges are subsets of the **real numbers**, we define the function f+g by $(f+g)(x)$ to be the function whose value at x is the sum of f(x) and g(x). Symbolically,

(f + g)(x) = f(x) + g(x). This is called pointwise addition of f and g.

The domain of **f+g** is the **intersection** of the domains of f and g since to compute $(f+g)(x)$ it is necessary and sufficient to compute both f(x) and g(x).

Other operations on functions are defined similarly:

- $(fg)(x) = f(x)g(x)$ (pointwise multiplication)
- $f^p(x) = (f(x))^p$ for any real exponent p with the domain of f^p consisting of those points for which the p-th power of f(x) makes sense.
- $(f/g)(x) = f(x)/g(x)$, for $g(x) \neq 0$ (pointwise multiplication)

For example, if $f(x) = 3 \sin(x)$ and $g(x) = x^2$, then

$(f+g)(x) = 3\sin(x) + x^2$
$(fg)(x) = 3\sin(x)*x2$
$(f-g)(x) = 3\sin(x) - x^2$
$(f/g)(x) = 3\sin(x) / x^2$

The domains of both f and g are all **real numbers**, but the domain of f/g is $\{x \mid x \neq 0\}$.

Now let us consider two functions f and g from A= {1,2} to, B= {1,2,3,4}, where f={(1,1),(2,4)}. Let g be defined by the rule $g(x)= x^2$ where the domain of g is the set {1,2}. Here both have the same domain. Since f and g assign the same image to each element in the domain, they have the same effect throughout. This is why we treat them as the same, or equal.

Definition: If f and g are two functions defined on the same domain A and if **f(a)= g(a)** for every $a \in A$, then the functions **f** and **g** are **equal**, i.e., f = g.

For example $f(x) = x^2+5$, where x is a real number, and $g(x) = x^2+5$, where x is a complex number. Then the function f is not equal to the function g since they have different domains although $f(x)= x^2+5= g(x \;\forall x \in \mathbf{R})$. By this example we can conclude that even if f(a)= g(a), f and g may not be the same.

So far, the operations you have seen are the same as those for member systems. However, there is yet another operation on functions which we now define.

Definition: Let f and g be the operation of combining two functions by applying them one after the other. That is, the composition of f(x) and g(x), denoted by, fog .

For example, consider f : **R**$\rightarrow$**R** : $f(x) = (x^3 + 2x)^3$. We can write it as the composition of g and h, where the value of f(x) can be obtained by first calculating $x^3 + 2x$ and then taking its third power . We can write g for first or inside function $g(x) = x^3 + 2x$. We write h for the second function : $h(x) = x^3$. The use of the variable x is irrelevant, we could as well write $h(y) = y^3$ for $y \in$ **R.** We can see that $goh(x) = g(x^3 + 2x) = (x^3 + 2x)^3 = f(x)$.

In general $(f o g) \neq (g o f)$.

For example, if, $f(x) = x^2$ and $g(x) = x+1$, then $(f o g)(x) = (x + 1)^2$ and $(g o f)(x) = x^2+1$.
Here we can see that $f o g \neq g o f$.

Let us see another example, where $f(x) = x^2$, $g(x) = x +1$, $h(x) = x^3$
Then, $f o (g o h)\ (x) = (x^3 +1)^2$ and $(f o g)\ o h\ (x) = (x^3+1)^2$. Here we can see $f o (g o h) = (f o g)\ o h$.

Now let us see how you can get **product** of two permutations f and g of the same set,

Let $f=\begin{pmatrix} a_1 & a_2 \ldots & a_n \\ f(a_1) & f(a_2) \ldots & f(a_n) \end{pmatrix}$ and

$g=\begin{pmatrix} a_1 & a_2 \ldots & a_n \\ g(a_1) & g(a_2) \ldots & g(a_n) \end{pmatrix}$. Then $fg=\begin{pmatrix} a_1 & a_2 \ldots & a_n \\ f[g(a_1)] & f[g(a_2)] \ldots & f[g(a_n)] \end{pmatrix}$ is itself a permutation.

For example if, $f=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $g=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ then

$fg=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ and $gf=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Note that $f o g \neq g o f$. Thus the **multiplication of permutations is not commutative** in general.

However, the multiplication of permutations is associative. For example, if $f=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, $g=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $h=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ be the permutations on A = {1,2,3,4}, then

$fg=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$, $gf=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$, $gh=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$,

$f(gh)=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $(fg)h=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$.

Here we can see the multiplication of permutation is commutative.

Now try these exercises.

E 29) Let $f(x) = 1/x$ and $g(x) = x^3 + 2$. Find the following functions, where $x \in$ **R**.

i) $(f + g)(x)$

ii) $(f - g)(x)$

iii) $(fg)(x)$

iv) $(f/g)(x)$

E30) Let $f(x) = \sqrt{x+1}$ $\forall x \geq -1$ and $g(x) = x^3$ $\forall x \in \mathbf{R}$. Define the following functions. Also give their domains.

i) $(f + g)$

ii) $(f - g)$

iii) (fg)

iv) (f/g)

v) $(f \circ g)$

With this we have come to the end of this unit. Let us now summaries what we have covered in this unit.

1.6 SUMMARY

In this unit we have covered the following points:

1. We introduced basic concepts related to sets and different ways of representing them.

2. We worked at different operations on sets and there Venn diagram representations.

3. We explored some properties common to operations on sets and logical statements.

4. In the process we also documented the duality principle.

5. We defined relations as a Cartesian product of sets and looked at several examples and type of relations.

6. We defined a function as a particular kind of relation. Then we studied different types of functions as well as basic operations on functions. In the process we considered permutations and their product.

1.7 SOLUTIONS / ANSWERS

E1) A={x:x is a student of IGNOU.}

E2) i) The collection of all good teachers is **not** a set because this collection is not well-defined. The characteristic 'good' cannot be measured objectively.

ii) The set of points on a line is **not** finite because infinitely many points make a straight line.

E3) $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

E4) Set of vowels of English alphabet V={a,e,i,o,u}. Two subsets of set V are V1={a,e}, and V2={i,o}. Two supersets of V are V3={a,b,c,…,z}, and V4={a,c,d,e,i,o,u,…,z}.

E5) Powerset of A={a,e,i,o,u} is

{ϕ,{a},{e},{i},{o},{u},{a,e},{a,i},{a,o},{a,u},{e,i},{e,o},{e,u},{i,o},{i,u},
{o,u},{a,e,i},{a,i,o},{a,o,u},{e,i,o},{i,o,u},{i,o,u},{a,e,i,o},{e,i,o,u},{a,e,i,o,u}.

E6) For empty set A={ } or ϕ, P(A)=1

E7) If $A \subseteq B$, then $P(A) \subseteq P(B)$ because every subset of A is a subset of B.

E8) If P(A) =P(B) then $A \in P(A) = P(B) = A \subseteq B$. Similarly, $B \subseteq A$. Therefore A = B.

E9)

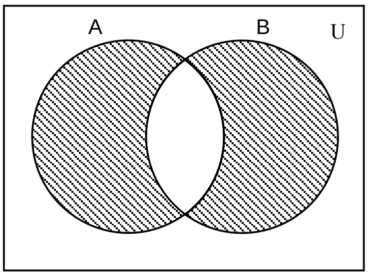


Fig.12: The Shaded portion is $A \Delta B$

You can try them for the other situations. We are showing in Fig. 12 for the second situation.

E10) $A \cup (B \cap C)$ = {Math, Physics, Science}= A.

E11) i) A ~B ={1,2,3}

ii) B ~A ={7,8,9}

iii) $A \Delta B$ ={1,2,3,7,8,9}

E12) Only if A and B are ϕ.

E13) Write separate functions to find A~B, B~A and $A \Delta B$ with passing sets A and B as argument, return the resultant set.

E14) $A \cap B$ can be equal to $A \cup B$ if either $A \subseteq B$ or $B \subseteq A$.

E15) i) Dual of $A \cap (B \cap C) = (A \cap B) \cap C$ is $A \cup (B \cup C) = (A \cup B) \cup C$.

ii) Dual of $(A \cup B) \cap (A \cup C)$ is $(A \cap B) \cup (A \cap C)$.

E16)

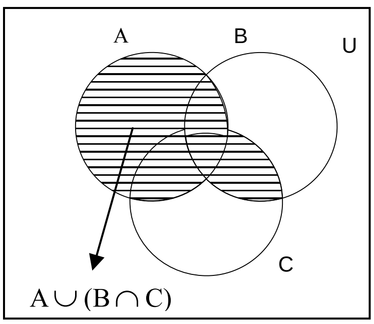


Fig.13: The lined portion represents $A \cup (B \cap C)$

E17)

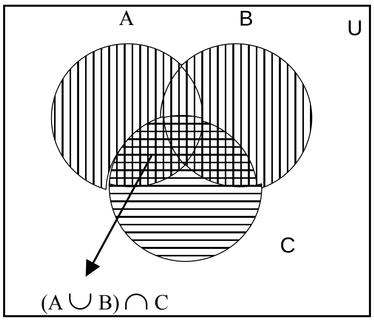


Fig.14(a)

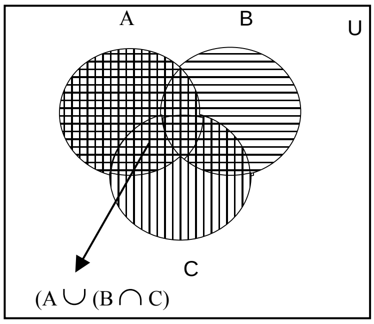


Fig.14(b)

Shaded area in Fig.14 (a) and Fig.14(b) are not same so $(A \cup B) \cap C$ is not equal to $(A \cup (B \cap C)$.

E18) i) X X X={(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)}.

ii) X X Y={(a,1), (b,1), (c,1), (a,2), (b,2), (c,2), (a,3), (b,3), (c,3)}.

iii) X X ϕ = ϕ.

E19) A X B = B X A iff A = B.

E20) The geometric diagram for RX{2} will be the line parallel to Y axis. See Fig.15.

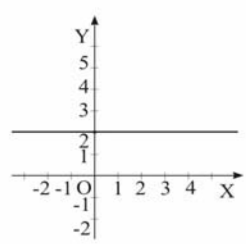


Fig.15: y=2

E21) i) For a $\in$ A, aRa is reflexive because every one loves herself or himself.

ii) R is not symmetric because if a loves b then b need not love a, i.e., aRb does not always imply bRa. Thus R is not symmetric.

iii) R is not transitive, because if a loves b and b loves c then a need not love c; i.e., if aRb and bRc, aRc need not be. Thus, R is not transitive.

Hence, R is reflexive but is neither symmetric nor transitive.

E22) R is a function because each element of A is assigned to a unique element of B.

E23) Not every relation is a function. For example, this relation does not satisfy the property that,

a) Each element of A must have assigned one element in B.

b) If $a \in A$ is assigned $b \in B$ and $a \in A$ is assigned $b' \in B$ then $b = b'$.

That is why relations those who don't satisfy above properties are not a function

E24) We can see that the code has no effect on the value of n $\leq$ 0.In the While loop, the value of n is halved whenever it is even. If n becomes odd before reaching 1, the second part of the while loop is invoked, and n remains odd and increases forever.

This shows that f : N→N is the function defined by $f(n)=\begin{cases} 0 \text{ if } n = 0. \\ 1, \text{if n is a power of } 2, \\ \text{undefined otherwise} \end{cases}$

E25) The domain of f is {1,2,3,4} and range of f is {2,3,4,5}.

E26) Function $f(x) = x^2$ is one-to-one because for every value of x, x^2 will be a number that is different for different x. Hence, $f(x) = x^2$ is one-one mapping.

E27) One-to-one function will be used for providing identity card number, because each person must have unique identity numbers.

E28) Step 1: $y = x^3 - 3$

Step 2: $x = y^3 - 3$

Step 3: $y = \sqrt[3]{x+3}$

Step 4: $f^{-1}(x) = \sqrt[3]{x+3}$.

E29) i) $(f+g)(x) = \frac{1}{x} + x^3 + 2$

ii) $(f-g)(x) = \frac{1}{x} - (x^3 + 2)$

iii) $(f.g)(x) = \left(\frac{1}{x}\right)(x^3 + 2)$

iv) $(f/g)(x) = \frac{1}{x(x^3+2)}$ $\forall x \in R$.

E30) i) $(f+g)(x) = \sqrt{x+1} + x^3$ $\forall x \geq -1$

ii) $(f\text{-}g)(x) = \sqrt{x+1} - x^3$ $\forall x \geq -1$

iii) $(f.g)(x) = \sqrt{x+1}\ x^3$ $\forall x \geq -1$

iv) $(f/g)(x) = \sqrt{x+1}\ /x^3$ $\forall x \geq -1, x \neq 0$

v) $(f o g)(x) = f(x^3) = \sqrt{x^3+1}$ $\forall x \geq -1$.

UNIT 2 COMBINATORICS — AN INTRODUCTION

Structure **Page No.**

2.0 INTRODUCTION

Let us start with thinking about how to assess the efficiency of a computer programme. For this we would need to estimate the number of times each procedure is called during the execution of the programme. How would we do this? The theory of combinatorics helps us in this matter, as you will see while studying this unit.

Combinatorics deals with counting the number of ways in which objects can be arranged according to some pattern (listing). Mostly, it deals with a finite number of objects and a finite number of ways of arranging them. Sometimes an infinite number of objects and infinite number of ways in which they can be arranged are also considered. However, in this unit and block, we shall restrict our discussion to a finite number of objects.

We start our discussion in Sec. 2.2, with two counting principles. These principles help us in counting the number of ways in which a task can be done when it consists of several subtasks, and there are many possible ways of doing the subtasks.

In Sec. 2.3 we look at arrangements of objects in which the order matters. Such arrangements are called permutations. Here we look at various linear and circular permutations, and how to count their number in a given situation.

In Sec. 2.4, we consider arrangements of objects in which the order does not matter. Such arrangements are called combinations. We will consider situations that require us to count combinations. You will see that most of these situations require us to apply the multiplication principle also.

In the next section, Sec. 2.5, we consider binomial and multinomial coefficients. We see how they are related to the objects studied in Sec. 2.4.

Finally, in Sec. 2.6, we consider the applications of what we have presented in the rest of the unit, for finding the probability of the occurrence of an event. As you will see, this application is natural, since we use similar counting arguments for obtaining discrete probabilities. This discussion will be useful for you, for instance, in coding theory as well as in designing **reliable** computer systems.

We continue our study of combinatorics in the next unit. We also have a section of miscellaneous exercises at the end of the block of which several are based on this unit. Doing these exercises, and every exercise given in the unit, will help you achieve the following objectives of this unit.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- explain the multiplication and addition principles, and apply them;
- differentiate between situations involving permutations and those involving combinations;
- perform calculations involving permutations and combinations;
- prove and use formulae involving binomial and multinomial coefficients;
- apply the concepts presented so far for calculating combinatorial probabilities.

2.2 MULTIPLICATION AND ADDITION PRINCIPLES

Let us start with considering the following situation: Suppose a shop sells six styles of pants. Each style is available in 8 lengths, six waist sizes, and four colours. How many different kinds of pants does the shop need to stock?

There are 6 possible types of pants; then for each type, there are 8 possible length sizes; for each of these, there are 6 possible waist sizes; and each of these is available in 4 different colours. So, if you sit down to count all the possibilities, you will find a huge number, and may even miss some out! However, if you apply the multiplication principle, you will have the answer in a jiffy!

So, what is the multiplication principle? There are various ways of explaining this principle. One way is the following:

Suppose that a task/procedure consists of a sequence of subtasks or steps, say, Subtask 1, Subtask 2,…, Subtask k. Furthermore, suppose that Subtask 1 can be performed in n_1, ways, Subtask 2 can be performed in n_2 ways after Subtask 1 has been performed, Subtask 3 can be performed in n_3 ways after Subtask 1 and Subtask 2 have been performed, and so on. Then **the multiplication principle** says that the number of ways in which the whole task can be performed is $n_1.n_2….n_k$.

Let us consider this principle in the context of boxes and objects filling them. Suppose there are m boxes. Suppose the first box can be filled up in k(1) ways. For every way of filling the first box, suppose there are k(2) ways of filling the second box. Then the two boxes can be filled up in k(1).k(2) ways. In general, if for every way of filling the first $(r – 1)$ boxes, the rth box can be filled up in k(r) ways, for r = 2,3,…, m, then the total number of ways of filling all the boxes is k(1).k(2)… k(m).

So let us see how the multiplication principle can be applied to the situation above (the shop selling pants). Here k(1) = 6, k(2) = 8, k(3) = 6 and k(4) = 4. So, the different kinds of pants are $6 \times 8 \times 6 \times 4 = 1152$ in number.

Let's consider one more example.

Example 1: Suppose we want to choose two persons from a party consisting of 35 members as president and vice-president. In how many ways can this be done?

Solution: Here, Subtask 1 is 'choosing a president'. This can be done in 35 ways. Subtask 2 is 'choosing a vice-president'. For each choice of president, we can choose the vice-president in 34 ways. Therefore, the total number of ways in which Subtasks 1 and 2 can be done is $35 \times 34 = 1190$.

* * *

There is another fundamental principle called the **addition principle**. This is applied in situations like the following one:

Suppose that a task consists of performing exactly one subtask from among a collection of disjoint (mutually exclusive) subtasks, say, Subtask 1, Subtask 2,…., Subtask k. (i.e., the task is performed if **either** Subtask 1 is performed, **or** Subtask 2,…, or Subtask k is performed.) Further, suppose that Subtask i can be performed in n_i ways, i = 1,2,…, k. Then, the number of ways in which the task can be performed is the sum $n_1+n_2+\ldots+n_k$.

Let us consider an example of its application.

Example 2: There are three political parties, P_1, P_2 and P_3. The party P_1 has 4 members, P_2 has 5 members and P_3 has 6 members in an assembly. Suppose we want to select two persons, both from the same party, to become president and vice-president. In how many ways can this be done?

Solution: From P_1, we can do the task in $4 \times 3 = 12$ ways, using the multiplication principle. From P_2, it can be done in $5 \times 4 = 20$ ways. From P_3 it can be done in $6 \times 5 = 30$ ways. So, by the addition principle, the number of ways of doing the task is $12 + 20 + 30 = 62$.

* * *

Though both these principles seem simple, quite a number of combinatorial enumerations can be done with them. For instance, what we see from Example 2 is that the addition principle helps us to count all possible arrangements grouped into mutually exclusive and exhaustive classes.

Why don't you try a few exercises that involve the use of these principles now?

E1) Give a situation related to computing in which the addition principle is used, and one in which the multiplication principle is used.

E2) Find the number of words of length 4, meaningful or not, made with the letters a,b,…, j.

E3) If n couples are at a dance, in how many ways can the men and women be paired for a single dance?

E4) How many integers between 100 and 999 consist of distinct even digits?

E5) Consider all the numbers between 100 and 999 that have distinct digits. How many of them are odd?

Let us now consider certain arrangements of objects, in which the order in which they are arranged matters.

2.3 PERMUTATIONS

Suppose we have 15 books that we want to arrange on a shelf. How many ways are there of doing it? Using the multiplication principle, you would say —

$15 \times 14 \times 13 \times \ldots.. 2 \times 1 = 15!$

n! denotes '**n factorial**', which means $n \times (n-1) \times . \times 2 \times 1$ for any $n \in \mathbf{N}$.)

Each of these arrangements of the books is a permutation of the books. Let us define this term formally.

Definition: An arrangement of a set of n objects **in a given order** is called a **permutation** of the objects (taken altogether at a time).

An **ordered** arrangement of the n objects, taking r at a time, (where $r \leq n$) is called **a permutation of the n objects taking r at a time**. The total number of such permutations is denoted by **P(n,r)**.

As an example, let us consider picking out books, three at a time, from the shelf of 15 books. The first book can be chosen in 15 ways, the next in 14 ways, and the third in 13 ways. So the multiplication principle tells us that the total number of permutations of the 15 books taken 3 at a time is $P(15,3) = 15 \times 14 \times 13$.

Other notations used for P(n,r) are ${}^nP_r$, P^n_r, ${}_nP_r$.

Again, consider the permutations of a,b,c,d, taken 2 at a time. These are ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc. (Note that ab and ba are considered different even though they consist of the same two objects.) Or, we can argue combinatorically as above: The first letter can be chosen in 4 ways, and then the next letter can be chosen. We can list out all the cases in 3 ways. So, the total number of permutations are $P(4,2) = 4 \times 3 = 12$.

Now, is there a formula for finding the value of P(n,r)? This is what the following theorem tells us.

Theorem 1: The number of permutations of n objects, taken r at a time, where $0 \leq r \leq n$, is given by $P(n,r) = \dfrac{n!}{(n-r)!}$

Consider r boxes arranged in a line. Choose one object out of n and place it in the first box. This can be done in n ways. Then from the remaining (n–1) objects choose one and place it in the second box. The first two boxes can be filled in n(n – 1) ways. We continue this operation till the rth box is filled. So, by the multiplication principle, the total number of ways of doing this is n(n – 1) (n – 2) …(n – r+1).

$$\begin{aligned} P(n,r) &= n(n-1)\ldots(n-r+1). \\ &= n(n-1)\ldots(\,(n-r+1)(\,(n-r)(\,(n-r-1)\ldots3.2.1 \\ &= (n-r)\ldots(\,(n-r-1)\,\ldots3.2.1 \\ &= n!/(n-r)! \end{aligned}$$

We define 0! = 1

Proof: In particular, Theorem 1 tells us that the number of permutation of n objects, taken all at a time, is given by

$P(n,n) = n!$

and $P(n, 0) = 1 \qquad \forall n \in \mathbf{N}$.

So, for example, by Theorem 1 we can find

$P(6,4) = 6.5.4.3 = 6!/(6-4)!$ And $P(6,0) = 1$.

Why don't you try some exercises now?

E6) If m and n are positive integers, show that $(m+n)! \geq m! + n!$.

E7) How many 3-digit numbers can be formed from the 6 digits 2,3,5,7,8,9 if repetitions are not allowed? How many of these numbers are less than 400? How many are even?

E8) How many ways are there to rank n candidates for the job of chief engineer? In how many rankings will Ms. Sheela be in the second place.

In defining the concept of permutation we assumed that the objects were distinguishable. What does this mean, and what happens if we remove this assumption? Let's see.

2.3.1 Permutation of Objects Not Necessarily Distinct

We have shown that there are P(n,r) ways to choose r objects from a set of n distinct objects and arrange them in linear order. Here we consider the same problem with the relaxed condition that some of the objects in the collection may not be distinguishable.

For example, we consider permutations of the letters of the word DISTINCT. Here there are 8 letters of which 2 are I, 2 are T, and three are 4 other different letters. To count the permutations in such a situation, we have the following result.

Theorem 2: Suppose there are n objects classified into k distinct types, with m_1 identical objects of the first type, m_2 identical objects of the second type,…, and m_k identical objects of the kth type, where $m_1+m_2+...+m_k = n$. Then the number of distinct arrangements of these n objects, denoted by P(n; m_1, m_2,..m_k) is $\frac{n!}{m_1!m_2!...m_k!}$.

Proof: Let x be the number of such permutations. If the objects of Type i are considered distinct, then they can be arranged amongst themselves in m_1! ways, where i = 1,2,…, k. Therefore, by the multiplication principle, the total number of permutations of these n distinct objects, taken all at a time, is $xm_1!m_2!\ldots m_k!$.

But this is precisely n! when there are n distinct objects.

Hence, $xm_1!m_2!\ldots m_k! = n!$, that is, $x = n!/m_1!m_2!\ldots m_k!$

So for example, this result tells us that the number of distinct 8 letter words, not necessarily meaningful, that we can make from the letter of the word "DISTINCT" is $\frac{8!}{2!2!1!1!1!1!} = 14.$

Here are some related exercises.

E9) How many permutations are there of the letters, taken all at a time, of the words (i) ASSESSES, (ii) PATTIVEERANPATTI?

E10) How many licence plates can be made if each should have 3 letters of the English alphabet with no letter repeated? What will be the answer if the letters can be repeated?

So far, we have considered permutations of objects as linear arrangements of objects; this means that we visualize arrangements of objects in a **line**. But there is a variant in which the objects are arranged along the circumference of a circle. Let us consider that now.

2.3.2 Circular Permutation

Consider an arrangement of 4 objects, a,b,c,d as in Fig. 1. We observe the objects in the clockwise direction. On the circumference there is no preferred origin, and hence the permutations abcd, bcda, cdab, dabc will look exactly alike. So, each linear

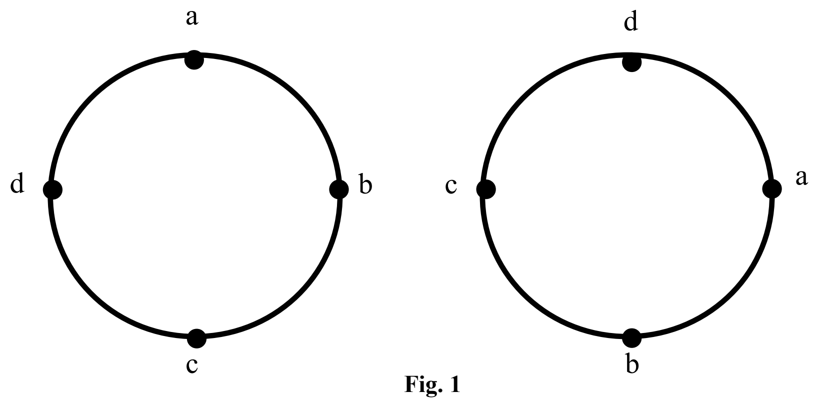


Fig. 1

permutation, when treated as a circular permutation, is repeated 4 times. Similarly, if n objects are placed in a circular arrangement, each linear arrangement is repeated n times. So, if we consider all the n! permutations of n things, each circular permutation will be indistinguishable from the (n−1) others obtained by the process of rotating the objects in the same order. So the number of distinct circular permutations will be n!/n = (n−1)!. Thus, we have shown that **the number of circular permutations of n things, taken all at a time, is (n−1)!.**

Let us consider some examples.

Example 3: In how many distinct ways is it possible to seat eight persons at a round table?

Solution: Clearly we need the number of circular permutations of 8 things. Hence the answer is 7! = 5040.

* * *

Example 4: In the preceding question, what would be the answer if a certain pair among the eight persons

(i) must not sit in adjacent seats?

(ii) must sit in adjacent seats

Solution: To answer (i), let us first solve (ii) from 7! we have to subtract the number of cases in which the pair of persons sit together. If we consider the pair as forming one unit, then we have the circular permutations of 7 objects, which is (7−1)! (Note that this is the answer for (ii).) But even as a unit they can be arranged in two ways. Hence the required answer is 2(6!). Now to answer (i), we must subtract these possibilities from the total number of ways of seating all the people. This is 7!−2(6!) = 3600.

* * *

Example 5: Suppose there are five married couples and they (10 people) are made to sit about a round table so that neither two men nor two women sit together. Find the number of such circular arrangements.

Solution: Five females can be made to sit about a round table in (5−1)! = 4! ways. One male can be seated in between two females. There are five positions, and hence they can be made to sit in 5! ways. By the multiplication principle, the total number of ways of such seating arrangements is 4! × 5! = 2880.

* * *

Example 6: Consider seven people seated about a round table. How many circular arrangements are possible if at least one of them will not have the same neighbours in any two arrangements?

Solution: The two distinct arrangements in Fig. 2 show that each has the same neighbours.

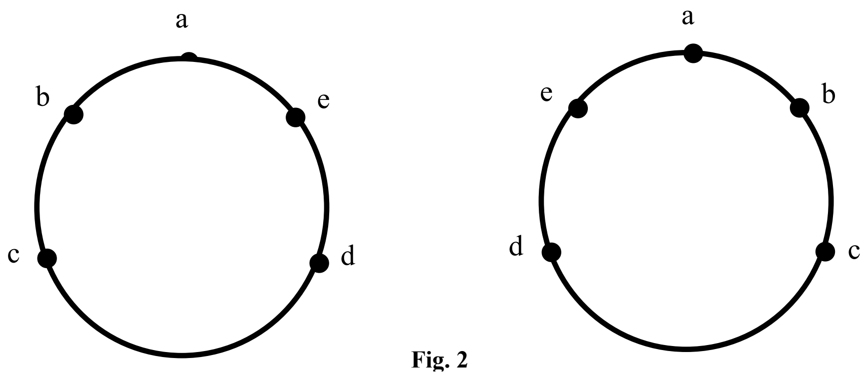


Fig. 2

Hence, the total number of circular arrangements = $(7-1)! \times \frac{1}{2} = 360$.

* * *

You may try the following exercise.

E11) If there are 7 men and 5 women, how many circular arrangements are possible in which women do not sit adjacent to each other?

Permutations apply to ordered arrangement of objects. What happens if order does not matter? Let's see.

2.4 COMBINATIONS

Let's begin by considering a situation where we want to choose a committee of 3 faculty members from a group of seven faculty members. In how many distinct ways can this be done? Here order doesn't matter, because choosing F_1, F_2, F_3 is the same as choosing F_2, F_1, F_3, and so on. (Here F_i denotes the ith faculty member.) So, for every choice of members, to avoid repetition, we have to divide by 3!. Thus, the number would be $\frac{7 \times 6 \times 5}{3!} = \frac{7!}{3!4!}$.

More generally, suppose there are n distinct objects and we want to select r objects, where $r \leq n$, where the order of **the objects in the selection does not matter**. This is called **a combination** of n things taken r at a time. The number of ways of doing this is represented by ${}_nC_r$, ${}^nC_r$, C^n_r, $\binom{n}{r}$ or C (n, r). We will use the notation C(n, r), in conformity with the notation P(n, r) for permutations. We read C(n, r) as 'n choose r' to emphasize the fact that only **choice** is involved but **not ordering**.

In the example that we started the section with, you saw that the number of combinations was 7!/3!4!, i.e., $\frac{P(7,3)}{3!}$. In fact, this relationship between C(n, r) and P(n, r) is true in general. We have the following result.

Theorem 3: The number of combinations of n objects, taken r at a time, where $0 \leq r \leq n$ is given by

$C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$.

Proof: C(n, r) counts the number of ways of choosing r out of n distinct objects without regard to the order. Any one of these choices is simply a subset of r objects of the set of n objects we have. Such a set can be ordered in r! ways. Thus, to each combination, there corresponds r! permutations. Hence there are r! times as many permutations as there are combinations. Hence, by the multiplication principle, we get

P(n, r) = r! C(n,r)

Therefore, $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$.

Using Theorem 3, we can very quickly find out, for instance, how many ways there are of choosing 2 rooms out of 20 rooms offered. This is $C(20,2) = \frac{20!}{18!2!} = 190$.

Now, to find C(20,2), I took a short cut. I cancelled 18! From the number and denominator. In practice, I only needed to calculate $\frac{20\times 19}{2\times 1}$. This practice is useful, in general, i.e., we use the identity $C(n, r) = \frac{n(n-1)\ldots r \text{ factors}}{r(r-1)\ldots r \text{ factors}}$ for calculations. In fact, sometimes r is much larger than n – r, in which case we cancel r!. This is also what the following result suggests.

Theorem 4: C(n, r) = C(n, n – r), for $0 \le r \le n$, $n \in \mathbf{N}$.

Proof 1: For every choice of r things from n things, there uniquely corresponds a choice of n – r things from those n objects, which are the unchosen objects. This one-to-one correspondence shows that these numbers must be the same. This proves the theorem.

Proof 2: $C(n, r) = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-n-r)!} = C(n, n-r)$.

Because of these two theorems we have, for instance,

C(n, n) = C(n, 0) = P(n, 0) = 1. C(n, 1), and = C(n, n–1) = P(n, 1) = n.

The numbers C(n, r) are also called the binomial coefficients as they occur as the coefficients of x^r in the expansion of $(1+x)^n$ in ascending powers of x, as you will see in Sec. 1.5. At this stage, let us consider some examples involving C(n, r).

Example 7: Evaluate C(6, 2), C(7, 4) and C(9, 3).

Solution: $C(6, 2) = \frac{6.5}{2.1} = 15, C(7,4) = \frac{7.6.5}{3.2.1} = 35, \text{and } C(9,3) = \frac{9.8.7}{3.2.1} = 84.$

Example 8: Find the number of distinct sets of 5 cards that can be dealt from a deck of 52 cards.

Solution: The order in which the cards are dealt is not important. So, the required number is $C(52, 5) = \frac{52!}{5!47!} = \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1} = 2{,}598{,}960$.

Example 9: Suppose a valid computer password consists of 8 characters, the first of which is the digit 1, 3 or 5. The rest of the 7 characters are either English alphabets or a digit. How many different passwords are possible?

Solution: Firstly, the initial character can be chosen in C(3, 1) ways. Now, there are 26 alphabets and 10 digits to choose the rest of the characters from, and repetition is allowed. So, the total number of possibilities for these characters is $(26+10)^7$.

Therefore, by the multiplication principle, the number of passwords possible are $C(3, 1).36^7$.

Here are some exercises now.

E12) At a certain office, a committee consisting of one male and one female worker is to be constituted from among 12 men and 15 women workers. In how many distinct ways can this be done?

E13) In how many ways can a prize winner choose any 3 CDs from the 'Ten Best' list?

E14) How many different 7-person committees can be formed, each containing 3 women and 4 men, from a set of 20 women and 30 men?

So far we have been considering combinations of distinct objects. Let us now look at combinations in which repetitions are allowed. We start with considering the following situations.

Suppose five friends stop at a sweet shop where each of them has one of the following: a samosa, a dosa, and a vada. The order of consumption does not matter. How many different purchases are possible?

Let s, t, and d represent samosa, dosa, vada, respectively. In the following table we have listed some possible ways of purchasing these. For instance, the second row represents the possibility that all 5 friends order only dosas.

s	d	**v**
x	x	xxx
xxx	xxxx	xx

These orders can also be represented by x's and $|$'s. For instance, the first row can be written as x$|$x$|$xxx. So, any order will consist of five x's and two $|$'s.

Conversely, any sequence of five x's and two $|$'s represents an order. So, there is a 1-to-1 correspondence between the orders placed and sequences of five x's and two $|$'s. But the number of such sequences is just the number of distinct ways of placing 2 $|$'s in 7 possible places. This is C(7,2).

More generally, if we wish to select with repetition, r out of n distinct objects, we are considering all arrangements of r of one kind (say x's) and $n - 1$ of the other kind (say $|$'s) (because $(n - 1)$ $|$'s are needed to separate n types).

The following result gives us the total number of such possibilities.

Theorem 5: Let n and r be natural numbers. Then the number of solutions in natural numbers, to the equation $x_1 + x_2 + \ldots + x_n = r$, is $C(n + r - 1,r)$. Equivalently, the number of ways to choose r objects from a collection of n objects, with repetition allowed, is $C(n + r - 1,r)$.

Proof: Any string will consist of r objects and $n - 1$ bars, to denote the n different categories in which these objects can fall. So, it will be a string of length $n + r - 1$, containing exactly r stars and $n - 1$ bars. The total number of such strings is the number of ways we can position $(n - 1)$ bars in r different places. This is $C(n + r - 1,r)$.

Now we demonstrate how such strings correspond to solution of the equation $x_1 + \ldots + x_n = r$.

$n - 1$ bars in the string divide the string into n substrings of stars. The number of stars in these n substrings are the values of $x_1, x_2, \ldots, x_n$. Since there are r stars altogether, the sum is r. Therefore, is a one-to-one correspondence between the strings and the solutions, and the theorem is proved.

Let us consider examples of the use of this result.

Example 10: In how many ways can a prize winner choose three books from a list of 10 best sellers, if repeats are allowed?

Solution: Here, note that a person can choose all three books to be the same title. Applying Theorem 5, the solution is $C(10 + 3 - 1, 3) = C(12, 3) = 220$.

* * *

Example 11: Determine the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 7$, where $x_i \geq 0$ for all $i = 1,2,3,4$.

Solution: The solution of the equation corresponds to a selection, with repetition, of size 7 from a collection of size 4. Hence, there are $C(4 + 7 - 1, 7) = 120$ solutions. ($n = 4$, $r = 7$ in Theorem 5.)

* * *

So, from this sub-section, we see the equivalence of the following:

(a) The number of integer solutions of the equation $x_1 + x_2 + \ldots + x_n = r$, $x_i \geq 0$, $1 \leq i \leq n$.

(b) The number of selections, with repetition, of size r from a collection of size n.

(c) The number of ways r identical objects can be distributed among n distinct containers.

Why don't you try some exercises now?

E15) A student in a college hostel is allowed four fruits per day. There are 6 different types of fruits from which she can choose what she wants. For how many days can a student make a different selection?

E16) An urn contains 15 balls, 8 of which are red and 7 are black. In how many ways can:

i) 5 balls be chosen so that all 5 are red?

ii) 7 balls be chosen so that at least 5 are red?

In this section we have considered choosing r objects, with repetition, out of n objects, regardless of order. What happens when order comes into the picture? Let's consider an example.

Example 12: A box contains 3 red, 3 blue and 4 white socks. In how many ways can 8 socks be pulled out of the box, one at a time, if order is important?

Solution: Let us first see what happens if order isn't important. In this case we count the number of solutions of $r+b+w = 8$, $0 \le r, b \le 3$, $0 \le w \le 4$. To apply Theorem 5, we write $x = 3 - r$, $y = 3 - b$, $z = 4 - 10$.

Then we have $x+y+z = 10 - 8 = 2$, and the number of solutions this has is $C(3+2-1,2) = 6$.

These 6 solutions are (1, 0, 1) (0, 1, 1), (1, 1, 0), (2, 0, 0), (0, 2, 0), (0, 0, 2). So, the corresponding solutions for (r, b, w) are

(3, 3, 2), (2, 3, 3), (3, 2, 3), (3, 1, 4), (2, 2, 4), (1, 3, 4).

Now, we consider order. From Theorem 2 we know that the number of ways of pulling out 3 red, 3 blue and 2 white socks in some order is $\frac{8!}{3!3!2!}$. This number would be the same if you had 2 red, 3 blue and 3 white socks, etc. By this reasoning and considering all different orderings, the number of possibilities is

$$3\left(\frac{8!}{3!3!2!}\right)+2\left(\frac{8!}{3!1!4!}\right)+\frac{8!}{2!2!4!}=3220.$$

* * *

What we see, via Example 13, is that if we want to find the number of possibilities wherein order matters and repetition is allowed them:

Step 1: Find the possibilities when order doesn't matter, using Theorem 5;

Step 2: Use Theorem 2, to find the possibilities for each solution obtained in Step 1.

Why don't you try and exercise now?

E17) How many 6-letter words, not necessarily meaningful can be formed from the letters of CARACAS?

Let us now consider why C(n,r) shows up as the coefficients in the binomial expansions.

2.5 BINOMIAL COEFFICIENTS

You must be familiar with expressions like a+b, p+q, x+y, all consisting of two terms. This is why they are called binomials. You also know that a **binomial expansion** refers to the expansion of a positive integral power of such a binomial. For instance, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ is a binomial expansion. Consider coefficients 1, 5, 10, 10, 5, 1 of this expansion. In particular, let us consider the coefficient 10, of a^3b^2 in this expansion. We can get this term by selecting a from 3 of the binomials and b from the remaining 2 binomials in the product (a+b) (a+b) (a+b) (a+b) (a+b). Now, a can be chosen in C(5, 3) ways, i.e., 10 ways. This is the way each coefficient arises in the expansion.

The same argument can be extended to get the coefficients of $a^r b^{n-r}$ in the expansion of $(a+b)^n$. From the n factors in $(a+b)^n$, we have to select r for a and the remaining (n – r) for b. This can be done in C(n, r) ways. Thus, the coefficient of $a^r b^{n-r}$ in the expansion of $(a+b)^n$ is C(n, r).

In view of the fact that C(n, r) = C(n, n – r), the coefficients of $a^r b^{n-r}$ and $a^{n-r} b^r$ will be the same. r can only take the values 0, 1, 2, …, n. We also see that C(n, 0) = C(n, n) = 1 are the coefficients of a^n and b^n. Hence we have established the binomial expansion.

$(a+b)^n = a^n + C(n, 1)\, a^{n-1}b + C(n, 2)a^{n-2}b^2 + \ldots + C(n, r)\, a^{n-r}b^r + \ldots + b^n$.

In analogy with 'binomial', which is a sum of two symbols, we have 'multinomial' which is a sum of two or more (though finite) distinct symbols. Multinomial expansion refers to the expansion of a positive integral power of a multinomial. Specifically we will consider the expansion of $(a_1 + a_2 + \ldots + a_m)^n$. For the expansion we can use the same technique as we use for the binomial expansion. We consider the nth power of the multinomial as the product of n factors, each of which is the same multinomial. Every term in the expansion can be obtained by picking one symbol from each factor and multiplying them. Clearly, any term will be of the form $a_1^{r_1} a_2^{r_2} \ldots a_m^{r_m}$ where $r_1, r_2, \ldots$, are non-negative integers such that $r_1+r_2+\ldots+ r_m = n$. Such a term is obtained by selecting a_1 from r_1 factors, a_2 from r_2 factors **from among the remaining ($n-r_1$) factors**, and so on. This can be done in

$C(n,r_1).\ C(n-r_1,r_2).C(n-r_1-r_2, r_3)\ldots C(n-r_1-r_2-\ldots-r_{m-1}, r_m)$ ways.

If you simplify this expression, it will reduce to $\dfrac{n!}{r_1!\,r_2!\ldots r_m!}$.

So, we see that the **multinomial expansion** is

$(a_1+a_2+\ldots+a_m)^n = \Sigma\ \dfrac{n!}{r_1!\,r_2!\ldots r_m!}\ a_1^{r_1} a_2^{r_2} \ldots a_m^{r_m}$

where the summation is over all non-negative integers $r_1, r_2, \ldots, r_m$ adding to n.

The coefficient of $a_1^{r_1} a_2^{r_2} \ldots a_m^{r_m}$ in the expansion of $(a_1+a_2+\ldots+ a_m)^n$ is $\dfrac{n!}{r_1!\,r_2!\ldots r_m!}$, and is called a **multinomial coefficient**, in analogy with the binomial coefficient. We represent this by $C(n; r_1, r_2, \ldots, r_m)$. This is also represented by many authors as $\left[\dfrac{n}{r_1, r_2, \ldots .. r_m}\right]$.

For instance, the coefficient of $x^2y^2z^2t^2u^2$ in the expansion of $(x + y + z + t + u)^{10}$ is $C(10; 2, 2, 2, 2, 2) = 10!/(2!)^5$.

Let us see an example involving such coefficients.

Example 13: What is the sum of the coefficients of all the terms in the expansion of $(a+b+c)^7$?

Solution: The required answer is $\sum \dfrac{7!}{r!s!t!}$, where the summation is over all non-negative integers r, s, t adding to n. But it is also the value of $\sum \dfrac{7!}{r!s!t!} a^r b^s c^t$ for a = b = c = 1.

So the answer is $(1 + 1 + 1)^7 = 3^7$.

* * *

This short detour was just to give you an idea of the way in which the Cs and Ps can be extended. Let us now consider some identities involving the binomial coefficients. We first consider Pascal's formula.

Theorem 6 (Pascal's formula): For all positive integers n and all r such that $1 \le r \le n$, $C(n + 1, r) = C(n, r) + C(n, r - 1)$.

Proof 1: The left hand side of the identity represents the number of ways of choosing r things out of (n+1) distinct things. Suppose we select an object from the (n+1) things and mark it. Then the number of combinations in which the marked thing is absent is C(n, r), as we then choose r things out of the unmarked n things. The number of combinations in which the marked thing is present is C(n, r–1), as we have to choose (r – 1), things out of the unmarked things, and attach the marked thing to it to make r things. Pascal's formula now follows from the fact that the sum of the last two numbers mentioned must be equal to C(n+1, r).

Proof 2: $C(n, r) + C(n, r - 1) = \dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r+1)!(r-1)!}$

$$= \frac{n!}{r!(n+1-r)!}(n - r + 1 + r) = C(n+1, r).$$

Pascal's formula gives us a recursive way to calculate the binomial coefficients, since it tells us the value of C(n, r) in terms of binomial coefficients with a smaller value of n. Note that we use the fact that C(n, 0) = 1 for all $n \ge 0$ to start the recursion, since Theorem 6 only applies for $1 \le r \le n$. This recursive approach allows us to form Pascal's triangle, the display of the binomial coefficients shown in Fig.4.

The nth row of Pascal's triangle gives the binomial coefficients C(n, r) as r goes from 0 (at the left) to n (at the right); the top row is Row D. This consists of just the number 1, for the case n = 0. The left and right borders are all 1's, reflecting the fact that C(n, 0) = C(n, n) = 1 for all n. Each entry in the interior of the Pascal's triangle is the sum of the two entries immediately above it to the left and right. We call this property the **Pascal property**. For example, each 15 in Row 6 (remember that we are starting the count of rows with 0) is the sum of the 10 and the 5 immediately above it.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1

Fig. 3: Pascal's triangle

The diagonals of Pascal's triangle are also interesting. The diagonal parallel to the left edge but moved one unit to the right reads (from the top down) 1, 2, 3, 4, 5,…, reflecting the fact that C(n, 1) = n for $n \ge 1$. The next diagonal to the right, reading 1, 3, 6, 10, 15,…, reflects the fact that differences increase by 1 as we move down the diagonal.

Let us now consider some identities involving binomial coefficients.

Identity 1: $C(n, 0) + C(n, 1) + C(n, 2) + \ldots + C(n, n-1) + C(n, n) = 2^n$

By setting a = b = 1 in the binomial expansion of $(a+b)^n$, we get this identity. In the context of sets, it tells us the number of distinct subset that can be formed from a set with n elements. Note that the number of subsets containing precisely r elements is C(n, r). Hence the total number of subsets is $\sum_{r=0}^{n} C(n,r) = 2^n$, by the identity. So, this identity tells us that **the number of distinct subsets of a set with n elements is 2^n.**

Identity 2: $C(n, 0) - C(n, 1) + C(n, 2) - \ldots + (-1)^n C(n, n) = 0$.

We get this by setting a = 1, b = –1 in the expansion of $(a+b)^n$.

Now, adding the two identities, we get

$$2 \sum_{r \text{ even}} C(n, r) = 2^n, \text{ i.e., } \sum_{r \text{ even}} C(n, r) = 2^{n-1}$$

Similarly subtracting the second identity from the first leads us to the equation

$$\sum_{r \text{ odd}} C(n, r) = 2^{n-1}.$$

These two equations tell us that the number of subsets of a set of n elements with an even number of elements is equal to the number of subsets with an odd number of elements, both being 2^{n-1}.

Why don't you try to prove some identities now?

E18) Show that $C(n, m)\, C(m, k) = C(n, k)\, C(n-k, m-k)$, $1 \le k \le m \le n$.

E19) Prove that $C(k, k) + C(k + 1, k) + C(k + 2, k) + \ldots + C(n, k) = C(n+1, k+1)$ for all natural numbers $k \le n$.

Before ending this section, we just mention another extension of the definition of binomial coefficients. So far, we have defined C(n, r) for $n \ge r \ge 0$. We can extend this definition for any real number x, and any non-negative integer k, by

$$C(x, k) = \frac{x(x-1)\ldots(x-k+1)}{k!}.$$

This definition coincides with that of C(n, k), when n is a non-negative integer.

So far, in this unit, we have considered various ways of counting different kinds of arrangements. These methods are, not surprisingly, helpful in finding the probability of an event. We shall now discuss this.

2.6 COMBINATORIAL PROBABILITY

Historically, counting problems have been closely associated with probability. The probability of getting at least 6 heads on 10 flips of a fair coin, the probability of finding a defective bulb in a sample of 25 bulbs if 5 percent of the bulbs from which the sample was drawn are defective — all these probabilities are essentially counting problems. In fact, Pascal's triangle (Fig. 4) was developed by Pascal around 1650 while analysing some gambling probabilities.

Let us start by recalling some basic facts about probability. An **experiment** is a clearly defined procedure that produces one of a given set of outcomes. The set of all outcomes is called **the sample space** of the experiment.

For example, the experiment could be checking the weather to see if it is raining or not on a particular day. The sample space here would be {raining, not raining}.

Given an experiment, we can often associate more than one sample space with it. For instance, suppose the experiment is the tossing of two coins.

i) If the observer wants to record the number of tails observed as the outcomes, the sample space is {0, 1, 2}.

ii) If the outcomes are the sequence of heads and tails observed, then the sample space is {HH, HT, TH, TT}.

A subset of the sample space of an experiment is called an **event**. For example, for an experiment consisting of tossing 2 coins, with sample space {HH, HT, TH, TT}, the event that two heads do **not** show up is the subset {HT, TH, TT}.

Suppose X is a sample space of an experiment with N out comes. Then, the events are all the 2^N subsets of X. The empty set ϕ is called the **impossible event**, and the set X itself is called the **sure event**.

Now, for the purpose of this course, we will assume that all the outcomes of an experiment are **equally likely**, that is, there is nothing to prefer one case over the other. For example, in the experiment of coin tossing, we assume that the coin is unbiased. This means that 'head' and 'tail' are equally likely in a toss. The toss itself is considered a random mechanism ensuring 'equally likely' outcomes. Of course, there are coins that are 'loaded', which means that one side of the coin may be heavier than the other. But such coins are excluded from our discussion. Also, in our discussions we shall always assume that our **sample space is finite**.

Given this background, we have the following definition.

We represent the number of elements of a finite set A, i.e., the **cardinality** of A, by n(A) or $|A|$.

Definition: Then the **probability of the event A**, represented by **P(A)**, is $\frac{n(A)}{n(X)}$.

For instance, the probability that a card selected from a desk of 52 cards is a spade is $\frac{13}{52}$, because A is the set of 13 spades in the deck.

From the definition, we get the following statements:

i) As n $(\phi) = 0$, it follows that $P(\phi) = 0$.

ii) By definition, $P(X) = \frac{n(X)}{n(X)} = 1$.

iii) If A and B are two events, then $n(A\cup B) = n(A) + n(B) - n(A\cap B)$. Therefore, $P(A\cup B) = P(A)+P(B) - P(A\cap B)$.

iv) (**Addition Theorem in Probability**) : If A and B are two mutually exclusive events, then the probability of their union is the sum of the probabilities of A and B. i.e., if $A\cap B = \phi$, then $\mathbf{P(A\cup B) = P(A) + P(B)}$.

[This is a consequence of (i) and (iii) above.]

v) Suppose A is an event. Then the probability of A^c (also denoted by A'), the event complementary to A, or the event 'not A' is $1 - P(A)$, i.e.,

$\mathbf{P(A^c) = 1 - P(A)}$.

The reason is that the events A and A^c are mutually exclusive and exhaustive, i.e., $A \cup A^c = X$ and $P(A) + P(A^c) = 1$.

vi) (The generalised addition theorem) : If the events $A_1, A_2, \ldots, A_m$ are pairwise disjoint (i.e., mutually exclusive), then $\mathbf{P(\bigcup_i A_i) = \sum_i P(A_i).}$

Let us consider some examples from combinatorial probability.

Example 14: A die is rolled once. What are the probabilities of the following events?

(i) getting an even number,

(ii) getting at least 2,

(iii) getting at most 2,

(iv) getting at least 10.

Solution: If we call the events A, B, C and D, then we have X = {1, 2, 3, 4, 5, 6}, A = {2,4,6}, B = {2,3,4,5,6}, C = {1,2}, and D = ϕ.

Hence, P(A) = 3/6, P(B) = 5/6, P(C) = 2/6, P(D) = 0.

* * *

Example 15: A coin is tossed n times. What is the probability of getting exactly r heads?

Solution: If H and T represent head and tail, respectively, then X consists of sequences of length n that can be formed using only the letters H and T. Therefore, $n(X) = 2^n$. The event A consists of those sequences in which there are precisely r Hs. So, n(A) = C(n, r). Hence, the required probability is $C(n, r)/2^n$.

* * *

Example 16: Two dice, one red and one white, are rolled. What is the probability that the white die turns up a smaller number than the red die?

Solution: If the number on the red die is x and that on the white die is y, then X consists of the 36 pairs (x, y), where x and y can be any integer from {1, 2, 3, 4, 5, 6}.

For the event A, we need x < y. For x = 1, 2, 3, 4, 5, y can be x + 1, x+2,…, 6, i.e., 6 – x in number. Thus, by the addition principle,

$$n(A) = \sum_{x=1}^{5} (6 - x) = 5 + 4 + 3 + 2 + 1 = 15.$$

Hence, P(A) = 15/36 = 5/12.

* * *

Example 17: If a five-digit number is chosen at random, what is the probability that the product of the digits is 20?

Solution: If X is the collection of all 5-digit numbers, then $n(X) = 9.10^4 = 90000$. Now, 20 can be factored in only two ways, viz., 1.1.1.4.5 and 1.1.2.2.5, as the product of five factors. Of course, these factors can be permuted to give all possible cases for A. The numbers 5, 4, 1, 1, 1 can be permuted in 5!/1!1!3! = 20 ways, and the numbers 5, 2, 2, 1, 1 can be permuted in 5!/1!2!2! = 30 ways.

So, n(A) = 20 + 30 = 50.

Hence, P(A) = 50/90000 = 1/1800.

* * *

Example 18: Suppose A and B are mutually exclusive events such that P(A) = 0.3 and P(B) = 0.4. What is the probability that

i) A does not occur?

ii) A or B occurs?

iii) Either A or B does not occur?

Solution:

i) This is $P(A^c) = 0.7$.

ii) This is $P(A \cup B) = 0.7$.

iii) This is $P(A^c \cup B^c) = P[(A \cap B)^c] = P(\phi^c) = P(X) = 1$

* * *

Try some exercises now.

E20) A,B,C and D are four candidates for a chairperson's post. Suppose that A is twice as likely to be elected as B, B is thrice as likely as C, and C and D are equally likely to be elected. What is the probability of election of each candidate?

E21) In a ten-question true-false exam, a student must achieve six correct answers to pass. If she selects her answers randomly, what is the probability that she will pass?

There are several other methods for solving combinatorial problems. These will be taken up in the next two units. Let us now summarise what we have covered in this unit.

2.7 SUMMARY

In this unit we have discussed some counting techniques. Specifically, we have covered the following points.

1. The multiplication and addition principles for counting the number of ways in which a task can be completed.
2. What a permutation is, the derivation of the formula $P(n,r) = \dfrac{n!}{(n-r)!}$, and its application for solving problems.
3. The number of distinct arrangement of n objects of which m_1 are of Type 1, m_2 are of Type 2,…, m_k are of Type k, where $m_1, m_2 + \ldots, m_k = n$, is $P(n; m_1, m_2, \ldots, m_k) = \dfrac{n!}{m_1!m_2!\ldots m_k!}$.
4. What a circular permutation is, and that the number of such permutations of n objects, taken all at a time, is (n–1)!
5. What a combination is, the derivation of the formula

$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)r!}$, and its application for solving problems.

6. The proof and applications of the fact that the number of ways of choosing r objects from a collection of n objects, with repetition allowed, is C(n+r–1,r).

7. Why C(n,r) is called a binomial coefficients, and its analogue for multinomials.

8. Some identities involving C(n,r), including Pascal's formula C(n+1,r) = C(n,r) + C(n,r–1).

9. The use of counting techniques for finding some discrete probabilities.

2.8 SOLUTIONS/ ANSWERS

E1) For instance, both principles are used to find the number of ways in which 17 files are stored if there are 3 storage locations of 1000 K each and 10 files are of 100 K, 5 of 200 K 2 of 500 K.

E2) Here we apply the multiplication principle. Each letter has 10 possibilities. Therefore, the total number of words is 10^4.

E3) Suppose we number the men as 1, 2, 3,…, n. Then the first man can be paired with any of the n women, the second can be paired with any from the remaining (n – 1) women, and so on. Hence, the number of ways of pairing is n(n – 1)… 1 = n!.

E4) By the multiplication principle, the number of integers between 100 and 999 with all digits even is 4.5.5 = 100 (Note that the first digit cannot be zero, but the second and third digits can be 0.)

E5) For a number to be odd the last digit should be odd. So, the last position can be filled up in 5 ways. If the middle position is filled up by 0, then the first position can be filled up in 8 ways. Thus the number of odd numbers with 0 in the middle position and all digits distinct is 40, by the multiplication principle.

If the middle position is filled up by a digit other than 0, then this can be done in 8 ways. Then the first position can be filled up in 7 ways. So, the number of odd numbers with all digits distinct with the middle digit not zero is 5.8.7 = 280.

Thus, by the addition principle the answer is 40 + 280 = 320.

E6) $(m + n)! = (m + n)\,(m + n - 1) \ldots (m + 1)\, m!$.
$\Rightarrow (m + n)! - m! = \geq m^n + n! \geq m!\,[n! + m^n - 1]$
$\Rightarrow (m + n)! - m! - n! \geq n!\,(m! - 1) + m!\,(m^n - 1) \geq 0$.

E7) Without repetitions, the number is P(6, 3). For the number to be less than 400, the leftmost digit can only be 2 or 3. The rest of the digits can be filled in P(5, 2) ways. So, the total number of numbers less than 400 will be 2P(5, 2). Similarly, the total number of even numbers is 3P(5, 2).

Note: That the addition principle has been used in both cases.

E8) A ranking is an ordering of the n candidates. This can be done in P(n,n) = n! ways. The total number of rankings in which Sheela is in 2nd place in P(n – 1, n – 1) = (n – 1)!

E9) In the word 'ASSESSES', we have A once, E twice, and S five times. Thus the number of permutations is 8!/1!2!5! = 168.
In the word 'PATTIVEERANPATTI', R, N and V occur once, P, E and I occur twice, A thrice and T four times. Thus the required number of permutations is 16!/1!1!1!2!2!2!3!4! = 9.10.

E10) By the multiplication principle, the answer is 26.25.24 if the letters cannot be repeated, and 26.26.26 if the letters can be repeated.

E11) The seven men can be seated first. This can be done in 6! ways. The women can sit in between two men. There are seven such places. So, the women can sit in P(7,5) ways. Hence the answer is 6! × P(7,5).

E12) This can be done in C(12, 1).C(15,1) ways, i.e., 180 ways.

E13) This can be done in C(10, 3) ways, i.e., 120 ways.

E14) The total number of possibilities is C(20,3).C(30,4) = 31,241,700.

E15) Applying Theorem 5, we get C(9, 4) = 126 days.

E16) i) Be careful! This is not an application of Theorem 5. This is only the number of ways of choosing 5 balls out of 8 balls, i.e. C(8, 5).

ii) First pick 5 red balls, in C(8,5) ways. Then pick the remaining 2 arbitrarily. These 2 can be chosen in C(2+2–1, 2) = 3 ways. So, the total number of ways is C(8, 5) × 3.

E17) We have 2Cs, 3As, 1R and 1S. If order is not a concern, we consider the solutions of

$c+a+r+s = 6, 0 \le c \le 2, \; 0 \le a \le 3, \; 0 \le r, s \le 1.$

We convert this to the equivalent problem

$x+y+z+t = 1$, where $x = 2 - c$, $y = 3 - a$, $r = 1 - z$, $s = 1 - t$,

The number of solutions of this is C(4 + 1 – 1, 1) = 4.

There are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1).

The corresponding solutions in (c, a, r, s) are (1, 3, 1, 1), (2, 2, 1, 1), (2, 3, 0, 1), (2, 3, 1, 0).

Now order becomes important to us. Applying Theorem 2, the required number is

$$\frac{6!}{1!3!1!1!} + \frac{6!}{2!2!1!1!} + 2\left(\frac{6!}{2!3!1!0!}\right) = 420.$$

E18) The left side counts the ways to select a group of m people chosen from a set of n people and then select a subset of k leaders, say, of this group of m. This can also be done by selecting the subset of k leaders from the set of n people first, and then selecting the remaining m – k members of the group from the remaining n – k people. The number of ways in which this can be done is given on the right hand side. Therefore, the identity.

You can also prove this algebraically.

E19) One can prove this by induction on the variable n. The base case is trivial, since if n = 0, then k = 0 as well, and the equation reduces to C(0, 0) = C(1, 1), which is true. The induction step is proved by Pascal's formula and the induction hypothesis.

E20) The relative weightages of A, B, C and D are 6.3, 1, 1, respectively. So, P(A) = $\frac{6}{11}$, $P(B) = \frac{3}{11}$, $P(C) = \frac{1}{11} = P(D)$.

E21) The answer is same as the probability of getting at least 6 heads in 10 tosses of a true coin. Hence, the answer is

$C(10, 6)/2^{10} + C(10, 7)/2^{10} + C(10, 8)/2^{10} + C(10, 9)/2^{10} + C(10, 10)/2^{10}$

$= (210 + 120 + 45 + 10 + 1)/1024 = 193/512.$

UNIT 3 SOME MORE COUNTING PRINCIPLES

Structure **Page No.**

3.0 INTRODUCTION

In this unit, we continue our discussion of the previous unit on combinatorial techniques. We particularly focus on two principles of counting – the pigeonhole principle and the principle of inclusion-exclusion.

In Sec. 3.2 you will see how obvious the pigeonhole principle is. Its proof is very simple, and amazingly, it has several useful applications. We shall also include some of these in this section.

In Sec. 3.3, we focus on the principle (or formula) of inclusion-exclusion. As you will see, this principle tells us how many elements do not fit into any of n categories. We prove this result and also give a generalisation. Following this, in Sec. 3.4 we give several important applications of inclusion-exclusion.

We shall continue our discussion on combinatorial techniques in the next unit.

3.1 OBJECTIVES

After studying this unit, you should be able to:

- prove the pigeonhole principle, and state the generalised pigeonhole principle;
- identify situations in which these principles apply, and solve related problems;
- prove the principle of inclusion-exclusion;
- apply inclusion-exclusion for counting the number of surjective functions, derangements and for finding discrete probability.

3.2 PIGEONHOLE PRINCIPLE

Let us start with considering a situation where we have 10 boxes and 11 objects to be placed in them. Wouldn't you agree that regardless of the way the objects are placed in the two boxes at least one box will have more than one object in it? On the face of it, this seems obvious. This is actually an application of the pigeonhole principle, which we now state.

Theorem 1 (The Pigeonhole Principle): Let there be n boxes and (n+1) objects. Then, under any assignment of objects to the boxes, there will always be a box with more than one object in it.

This can be reworded as: if m pigeons occupy n pigeonholes, where m > n, then there is at least one pigeonhole with two or more pigeons in it.

Proof: Let us label the n pigeonholes 1, 2, …, n, and the m pigeons $p_1, p_2, \ldots, p_m$. Now, beginning with p_1, we assign one each of these pigeons the holes numbered 1, …, n, respectively. Under this assignment, each hole has one pigeon, but there are still (m–n) pigeons left. So, in whichever way we place these pigeons, at least one hole will have more than one pigeon in it. This completes the proof!

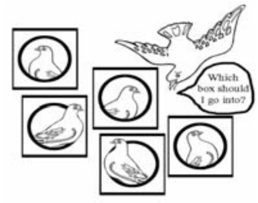


Fig. 1 The pigeonhole principle

This result appears very trivial, but has many applications. For example, using it you can show that:

- if 8 people are picked in any way from a group, at least 2 of them will have been born on the same weekday.
- in any group of 13 people, at least two are born in the same month.

Let us consider some examples of its application, in detail.

Example 1: Assuming that friendship is mutual, show that in any group of people we can always find two persons with the same number of friends in the group.

Solution: If there are n persons in the group, then let the number of friends the ith person has be f(i), i = 1,…, n. Clearly, f(i) can take values only between 0 and (n – 1).

If some f(i) is 0, it means that the ith person does not have any friends in the group. In this case, no person can be friends with all the other (n – 1) people. So, no f(i) can be (n – 1). So, only one of the values 0 or (n – 1) can be present among the f(i)'s. So, the n f(i)'s can take only (n – 1) distinct values. Therefore, by the pigeonhole principle, two f(i)'s must be equal. Then the corresponding i's have the same number of friends in the group.

* * *

Example 2: Suppose 5 points are chosen at random within or on the boundary of an equilateral triangle of side 1 metre. Show that we can find two points at a distance of at most ½ metre.

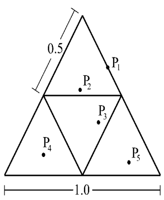


Fig. 2

Solution: Divide the triangle into four equilateral triangles of side ½ m by joining the midpoints of the sides by three line segments (see Fig. 2). These four triangles may now be considered as boxes and the five points as objects. By the pigeonhole principle, at least one of these smaller triangles will have two points in or on it. Clearly, the distance between these two points is at most ½ metre.

* * *

Example 3: Given any ten different positive integers less than 107, show that there will be two disjoint subsets with the same sum.

Solution: The highest numbers we could be given would be 97, 98, …, 106, which add up to 1015. So, consider pigeonholes marked 0, 1, 2,…, 1015. The set of 10 positive integers have $2^{10} = 1024$ subsets. Put a subset in the pigeonhole marked with the sum of the numbers in the set. The 1024 subsets have to be put in 1016 pigeonholes. So, some pigeonhole will have more than one subset with the same sum.

Now, note that two subsets that we get with the same sum, may not be disjoint. But, by dropping the common elements in them, we are left with disjoint subsets with the same sum.

* * *

Here are some related exercises for you to do.

E1) If 10 points are chosen in an equilateral triangle of side 3 cms., show that we can find two points at a distance of at most 1 cm.

E2) On 11 occasions a pair of persons from a group of 5 was called for a function. Show that some pair of persons must have attended the function at least twice.

E3) Four persons were found in a queue, independently, on 25 occasions. Show that at least on two occasions they must have been in the queue in the same order.

As you know, **mathematics develops through a process of generalisation**. You know that the principle is valid for n+1 objects and n boxes. It is natural to ask: what if we have, say, 4n+1 objects and 4 boxes? Can we prove a similar principle? In fact, we can, as given below.

Theorem 2 (The Generalized Pigeonhole Principle): If nm + 1 objects are distributed among m boxes, then at least one box will contain more than n objects.

This can be reworded as: Let k and n be positive integers. If k balls are put into n boxes, then some box contains at least [k/n] + 1 balls, where [x] denotes the greatest integer less than x.

Proof: We prove this by contradiction (see Unit 2, Block 1). Suppose all the m boxes have at most n objects in them. Then the total number of objects is at most nm, a contradiction. Hence, the theorem.

Applying this result, we see, for example, that suppose 479 students are enrolled in the course Discrete Mathematics, consisting of 6 units. Then, at least $\left[\dfrac{479}{6}\right]+1=80$ students are studying the same unit at a given point of time.

Let us consider a few more examples of the application of this principle.

Example 4: Show that in any group of 30 people, we can always find 5 people who were born on the same day of the week.

Solution: 30 people can be assigned to 7 days of the week. Then at least $\left[\dfrac{30}{7}\right]+1=5$ of them must have been born on the same day.

* * *

Example 5: 20 cards, numbered from 1 to 20, are placed face down on a table. 12 cards are selected one at a time and turned over. If two of the cards add up to 21, the player loses. Is it possible to win this game?

Solution: The pairs that can add up to 21 are (1, 20), (2, 19),…, (10,11). So, there are 10 such pairs. In turning 12 cards, at least one of these pairs will be included. Therefore, the player will lose.

* * *

Example 6: Show that every sequence of $n^2 + 1$ distinct integers includes either an increasing subsequence of n + 1 numbers or a decreasing subsequence of n + 1 numbers.

Solution: Let the sequence be $a_1, a_2, \ldots, a_{n^2+1}$. Suppose there is no increasing subsequence of n + 1 numbers. For each of these a_ks, let s(k) be the length of the longest increasing subsequence beginning at a_k. Since all n^2+1 of the s(k)'s are between 1 and n, at least $\left[\frac{n^2+1}{n}\right]+1 = n+1$ of these numbers are the same. (The s(k)'s are the objects and the numbers from 1 to n are the boxes.)

Now, if i < j and s(i) = s(j), then $a_i > a_j$. Otherwise a_i followed by the longest increasing subsequence starting at a_j would be an increasing subsequence of length s(j) + 1 starting at a_i. This is a contradiction, since s(i) = s(j).

Therefore, all the n + 1 integers a_k, for which s(k) = m, must form a decreasing subsequence of length at least n + 1.

* * *

Example 7: Take n integers, not necessarily distinct. Show that the sum of some of these numbers is a multiple of n.

Solution: Let S(m) be the sum of the first m of these numbers. If for some r and m, r < m, S(m) – S(r) is divisible by n, then $a_{r+1} + a_{r+2} + \ldots + a_m$ is a multiple of n. This also means that S(r) and S(m) leave the same remainder when divided by n. So, if we cannot find such pairs m and r, then it means that the n numbers S(1), S(2),…, S(n) leave different remainders when divided by n. But there are only n possible remainders, viz., 0, 1, 2,…, (n – 1). So, one of these numbers must leave a remainder of 0. This means that one of the S(i) s is divisible by n. This completes the proof.

In fact, in this example we have proved that one of the sums of consecutive terms is divisible by n.

* * *

You may like to try some exercises now.

E4) If any set of 11 integers is chosen from 1, …,20, show that we can find among them two numbers such that one divides the other.

E5) If 100 balls are placed in 15 boxes, show that two of the boxes must have the same number of balls.

E6) If $a_1, a_2, \ldots, a_n$ is a permutation of 1, 2,…, n and n is odd, show that the product $(a_1 - 1)(a_2 - 2)\ldots(a_n - n)$ must be even.

There are several corollaries to Theorem 2. We shall present one of them here.

Theorem 3: If a finite set S is partitioned into s subsets, then at least one of the subsets has $\frac{|S|}{k}$ or more elements.

Proof: Let $A_1, \ldots, A_k$ be a partition of S. (This means that $A_i \cap A_j = \phi$ for $i \neq j$ and $S = A_1 \cup A_2 \cup \ldots \cup A_k$.) Then the average value of $|A_i|$ is $\frac{1}{k}[\,|A_1| + \ldots + |A_k|\,] = \frac{|S|}{k}$.

So the largest A_i has at least $\frac{|S|}{k}$ elements.

A consequence of this result is the following theorem.

Theorem 4: Consider a function f: S $\to$ T, where S and T are finite sets satisfying $|S| > r|T|$. Then at least one of the sets $f^{-1}(t)$, $t \in T$, has more than r elements. ($f^{-1}(t)$ denotes the inverse image of the set {t}, i.e., $f^{-1}(t) = \{x \in S : f(x) = t\}$.)

Proof: The family $\{f^{-1}(t): t \in T\}$ partitions S into k sets with $k \le |T|$. By Theorem 3, some set in this family, say $f^{-1}(t')$, has at least $\frac{|S|}{k}$ members. Since $\frac{|S|}{k} \ge \frac{|S|}{T} > r$ by our hypothesis, $f^{-1}(t')$ has more than r elements.

Corollary: If f: S $\to$ T and $|S| > |T|$, then **f is not injective.**

Proof: Putting r = 1 in Theorem 4, we see that at least one of the sets $f^{-1}(t)$ has more than one element.

We conclude this section with some more extensions of the pigeonhole principle.

Theorem 5: Suppose we put an infinity of objects in a finite number of boxes. Then at least one box must have an infinity of objects.

Proof: If every box contains only a finite number of objects, then the total number of objects must be finite. Hence the theorem.

Theorem 6 (A generalisation of Theorem 3): Let $A_1, A_2, \ldots, A_k$ be subsets of a finite set S such that each element of S is in at least t of the sets A_i. Then the average number of elements in the A_is is at least $t.\frac{|S|}{k}$. (Note that, in this statement, the sets A_i may overlap.)

We leave the proof to you to do, and give you some related exercises now.

E7) Every positive integer is given one of the seven colours in VIBGYOR. Show that at least one of the colours must have been used infinitely many times.

E8) Let A be a fixed 10-element subset of {1, 2,…, 50}. Show that A possesses two different 5-element subsets, the sum of whose elements are equal.

E9) The positive integers are grouped into 100 sets. Show that at least one of the sets has an infinity of even numbers. Is it necessary that at least one set should have an infinity of even numbers and an infinity of odd numbers?

Let us now consider another very important counting principle.

3.3 INCLUSION-EXCLUSION PRINCIPLE

Let us begin with considering the following situation: In a sports club with 54 members, 34 play tennis, 22 play golf, and 10 play both. There are 11 members who play handball, of whom 6 play tennis also, 4 play golf also, and 2 play both tennis and golf. How many play none of the three sports?

To answer this, let S represent the set of all members of the club. Let T represent the set of tennis playing members, G represent the set of golf playing members, and H represent the set of handball playing members. Let us represent the number of elements in A by $|A|$. Consider the number $|S| - |T| - |G| - |H|$. Is this the answer to the problem? No, because those who are in T as well as G have been subtracted twice. To compensate for this double subtraction, we may now consider the number $|S| - |T| - |G| - |H| + |T \cap G| + |G \cap H| + |H \cap T|$. Is this the answer? No, because those playing all the three games have been subtracted thrice and then added thrice. But those members have to be totally excluded also. Hence, we now consider the number $|S| - |T| - |G| - |H| + |T \cap G| + |G \cap H| + |H \cap T| - |T \cap G \cap H|$. This is the correct answer. This reduces to 54 – 34 – 22 – 11 + 10 + 6 + 4 – 2 = 5.

To solve this problem we have made inclusions and exclusions alternately to arrive at the correct answer. This is a simple case of **the principle of inclusion and exclusion**. It is also known as the **sieve principle** because we subject the objects to sieves of a progressively finer mesh to arrive at a certain grading.

Let us state and prove this principle now.

A^c, or $\bar{A}$, denotes the complement of the set A

Theorem 7 (The inclusion-exclusion formula): Let $A_1, A_2, \ldots, A_n$ be n sets in a universal set U consisting of N elements. Let S_k denote the sum of the sizes of all the sets formed by intersecting k of the A_is at a time. Then the number of elements in none of the sets $A_1, A_2, \ldots, A_n$ is given by

$$\left|\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n\right| = N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n.$$

RHS is short for 'right-hand side'.

Proof: The proof is on the same lines of the counting argument given in the 'sports club' example at the beginning of this section. If an element is in none of the A_is, then it should be counted only once, as part of 'N' in the RHS of the formula above. It is not counted in any of the S_ks since it is in none of the A_is.

Next, an element in exactly one A_i, say A_r, is counted once in N, and once in S_1, and in none of the other S_ks. So the net count is 1–1 = 0.

Finally, take an element x in exactly m of the A_is. This is counted once in N, m times in S_1, C(m, 2) times in S_2 (since x is in C(m, 2) intersections $A_i \cap A_j$),…, C(m, k) times in S_k for $k \leq m$. x is not counted in any S_k for $k > m$. So the net count of x in the RHS of the formula is

$1 - C(m,1) + C(m,2) - \ldots + (-1)^k C(m, k) + \ldots + (-1)^m C(m, m) = 0$, by Identity 2 in Sec. 2.5.

So the only elements that have a net count of 1 in the RHS are those in $\bigcap_{i=1}^{n} \bar{A}_i$. The rest have a net count of 0. Hence the formula.

From this result, we immediately get the following one.

Corollary: Given the situation of Theorem 7,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = S_1 - S_2 + \ldots + (-1)^{k-1} S_k + \ldots + (-1)^{n-1} S_n.$$

Why don't you try and prove this result? (see E 10.)

What the inclusion-exclusion principle tells us is that to calculate the size of $A_1 \cup A_2 \cup \ldots \cup A_n$, calculate the size of all possible intersections of the sets $A_1, A_2, \ldots, A_n$. Add the results obtained by intersecting an odd number of the sets, and then subtract the results obtained by intersecting an even number of the sets. Therefore, this principle is ideally suited to situations in which

i) we just want the size of $A_1 \cup A_2 \cup \ldots \cup A_n$, not a listing of its elements, and

ii) multiple intersections are fairly easy to count.

Now let us consider some examples in which Theorem 7 is applied.

Example 8: How many ways are there to distribute r distinct objects into five (distinct) boxes with

i) at least one empty box?

ii) no empty box ($r \geq 5$)?

Solution: Let U be all possible distributions of r distinct objects into five boxes. Let A_i denote the set of possible distributions with the ith box being empty.

i) Then the required number of distributions with at least one empty box is $|A_1 \cup A_2 \cup \ldots \cup A_5|$. We have $N = 5^r$. Also, $|A_i| = (5-1)^r$, the number of distributions in which the objects are put into one of the remaining four boxes. Similarly, $|A_i \cap A_j| = (5-2)^r$, and so forth. Thus, by the corollary above, we have

$$|A_1 \cup \ldots \cup A_5| = S_1 - S_2 + S_3 - S_4 + S_5$$
$$= C(5,1)4^r - C(5,2)3^r + C(5,3)2^r - C(5,4)1^r + 0$$

ii) $\left|\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_5\right| = 5^r - C(5,1)4^r + C(5,2)3^r - C(5,3)2^r + C(5,4)1^r$, by Theorem 7.

* * *

Example 9: How many solutions are there to the equation x + y + z + w = 20, where x, y, z, w are positive integers such that $x \leq 6$, $y \leq 7$, $z \leq 8$, $w \leq 9$?

Solution: To use inclusion-exclusion, we let the objects be the solutions (in positive integers) of the given equation. A solution is in A_1 if $x > 6$, in A_2 if $y > 7$, in A_3 if $z > 8$, and in A_4 if $w > 9$. Then what we need is $\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4\right|$.

Now, to find the total number of **positive** solutions to the given equation, we rewrite it as $x_1+y_1+z_1+w_1 = 16$, where $x_1 = x+1$, $y_1 = y+1$, $z_1 = z+1$, $w_1 = w+1$. Any non-negative solution of this equation will be a positive solution of the given equation. So, the number of positive solutions is

N = C(16+4–1, 16) (see Example 11 of Unit 2)
= C(19, 3).

Similarly, $|A_1| = C(13,3), |A_2| = C(12,3), |A_3| = C(11,3),$

$|A_4| = C(10,3)$, $|A_1 \cap A_2| = C(6,3), |A_2 \cap A_3| = C(5,3)$, and so on. **Note** that for a solution to be in 3 or more A_i s, the sum of the respective variables would exceed 20, which is not possible. By inclusion- exclusion, we obtain

$$\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4\right| = C(19, 3) - C(13, 3) - C(12, 3) - C(11, 3) - C(10, 3)$$
$$+ C(6, 3) + C(5, 3) + C(4, 3) + C(4, 3) + C(3, 3) = 217.$$

* * *

Now you may try the following exercises.

E10) Prove the corollary to Theorem 7.

E11) How many numbers from 0 to 999 are not divisible by either 5 or 7?

Let us now consider applications of the inclusion-exclusion principle to some specific problem types.

3.4 APPLICATIONS OF INCLUSION-EXCLUSION

In this section we shall consider three broad kinds of applications — for counting the number of surjective functions, finding probability and finding the number of derangements.

3.4.1 Application to Surjective Functions

Let us first recall that a function $f : S \to T$ is called **surjective** (or **onto**) if $f(S) = T$, that is, if for every $t \in T \; \exists \, s \in S$ such that $f(s) = t$. Now let us prove a very useful result regarding the number of such functions.

Theorem 8: The number of functions from an m-element set **onto** a k-element set is $\sum_{i=0}^{k} (-1)^i C(k, i)(k-i)^m$, where $1 \le k \le m$.

Proof: We will use the inclusion-exclusion principle to prove this. For this, we define the objects to be all the functions (not just the onto functions) from M, an m-element set, to K, a k-element set. For these objects, we will define A_i to be the set of all f: $M \to K$ for which the ith element of K is not in f(M). Then what we want is $\left|\bar{A}_1 \cap .... \cap \bar{A}_k\right|$.

Now, the total number of functions from M to K is k^m. Also, the number of mappings that exclude a specific set of i elements in K is $(k - i)^m$, and there are C(k, i) such sets. Therefore, $|A_i| = (k-1)^m, |A_i \cap A_j| = (k-2)^m$, and so on.

Now, applying Theorem 7, we get

$$\left|\bar{A}_1 \cap ... \cap \bar{A}_k\right| = k^m - C(k,1)(k-1)^m + C(k,2)(k-2)^m - ... + (-1)^{k-1} C(k, k-1)1^m$$

Hence the result.

For example, the number of functions from a five-element set onto a three-element set are $\sum_{i=0}^{k} (-1)^i C(k,i)(k-i)^m$ for m = 5 and k = 3, that is, $3^5 - 3.2^5 + 3.1^5 = 150$.

Why don't you try some exercises now?

E12) Eight people enter an elevator. At each of four floors it stops, and at least one person leaves the elevator. After four floors the elevator is empty. In how many ways can this happen?

E13) How many six-digit numbers contain exactly three different digits?

Now we look at another application.

3.4.2 Application to Probability

An important application of the principle of inclusion-exclusion is used in probability. We have the following theorem.

Theorem 9: Suppose $A_1, A_2, \ldots, A_n$ are n events in a probability space. Then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r})$$

Proof: Let us begin by observing that $A_1 \cup A_2 \cup \ldots \cup A_n$ means that at least one of the events $A_1, A_2, \ldots, A_n$ occurs. Now, let the ith property be that an outcome belongs to the event A_i. By De Morgan's law, $\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n$ is the complement of $A_1 \cup A_2 \cup \ldots \cup A_n$. But the principle of inclusion-exclusion gives

$$\left| \overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n \right| = N - \sum (-1)^r \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left| A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r} \right|,$$ where N is the total number of outcomes.

Now, we divide throughout by N and note that

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n)$, to get the result.

Let us consider an example of the use of this result.

Example 12: Find the probability of a student in a college studying Japanese, given the following data:

All students have to study at least one language out of Hindi, Spanish and Japanese. 65 study Hindi, 45 study Spanish and 42 study Japanese. Further, 20 study Hindi and Spanish, 25 study Hindi and Japanese, 15 study Spanish and Japanese, and 8 study all 3 languages.

Solution: The total number of students is $|H \cup S \cup J|$, where H, S and J denote the number of students studying Hindi, Spanish and Japanese, respectively. By the inclusion-exclusion principle,

$$\begin{aligned} |H \cup S \cup J| &= |H| + |S| + |J| - |H \cap S| - |H \cap J| - |S \cap J| + |H \cap S \cap J| \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Therefore, the required probability is $\dfrac{|J|}{100} = 0.42$.

* * *

You could do the following exercises now.

E14) What is the probability that a 13-card hand has at least one card in each suit?

E15) What is the probability that a number between 1 and 10,000 is divisible by neither 2, 3, 5 nor 7?

Let us now come to the use of inclusion-exclusion for counting the number of a particular kind of permutation.

3.4.3 Application to Derangements

As you know, a permutation of a set is an arrangement of the elements of a set.

So, for example, a rearrangement $1 \to 1$, $2 \to 2$, $4 \to 3$, $3 \to 4$ is a permutation of the 4-element set {1, 2, 3, 4}. In this permutation, the position of the elements 1 and 2 are **fixed**, but the positions of 3 and 4 have been interchanged. Now consider the permutation $1 \to 4$, $2 \to 1$, $3 \to 2$, $4 \to 3$, of {1, 2, 3, 4}, in which the position of **every element** has been changed. This is an example of a derangement, a term we shall now define.

Definition: A **derangement** of a set S is a permutation of the elements of S which does not fix any element of S, i.e., it is a rearrangement of the elements of S in which the position of every element is altered.

So, if we treat a permutation as a 1-to-1 function from S to S, then a derangement is a function $f:S \to S$ such that $f(s) \neq s \ \forall \ s \in S$.

We have the following theorem regarding the number of derangements.

Theorem 10: The number of derangements of an n-element set is $D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$.

Proof: Let A_i be the set of all permutations of the n-element set that fix i $\forall$ i = 1, …, n. Then

$$D_n = \left| \bigcap_{i=1}^{n} \bar{A}_i \right| = n! - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \ldots + (-1)^n |A_1 \cap \ldots \cap A_n|$$

$$= n! - C(n, 1)\,(n-1)! + C(n, 2)\,(n-2)! - \ldots + (-1)^n C(n, n) 0!$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right),$$ which is the expression we wanted.

Remark: The expression $\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$ is the beginning of the expansion for e^{-1}. Even for moderately large values of n, D_n is very close to $n!e^{-1} = 0.36788\, n!$.

As an extension of Theorem 10, we have the following results.

Theorem 11: For a set of n objects, the number of permutations in which

(i) only r of these n objects are deranged is

$n! - C(r, 1)\,(n-1)! + C(r, 2)\,(n-2)! - \ldots + (-1)^r C(r, r)\,(n-r)!$;

(ii) exactly r elements are fixed is $C(n, r)\, D_{n-r}$.

We will not prove these formulae here, but shall consider some examples of their applications.

Example 12: Let n books be distributed to n children. The books are returned and distributed to the children again later on. In how many ways can the books be distributed so that no child will get the same book twice?

Solution: The required number is $(n!)^2e^{-1}$, since corresponding to each first distribution, there are $(n!)e^{-1}$ ways of distributing again.

* * *

Example 13: Suppose 10 people have exactly the same briefcases, which they leave at a counter. The cases are handed back to the people randomly. What is the probability that no one gets the right case?

Solution: The number of possibilities favourable to the event is D_{10}. The total number of possibilities is 10!. Thus, the probability that none will get the right briefcase is $D_{10}/10! = 0.36788$.

* * *

Note that, since $D_n \approx n!e^{-1}$, the possibility in all such examples is essentially e^{-1}, which is independent of n.

You may now try the following exercises.

E16) Each of the n guests at a party puts on a coat when s/he leaves. None of them gets the correct coat. In how many ways can this happen? In how many ways can just one of the guests get the right coat?

E17) In how many ways can the integers 1, 2, 3,…,9 be permuted such that no odd integer will be in its natural position.

E18) Find the number of permutations in which exactly four of the nine integers 1, 2,…,9 are fixed.

With this we come to the end of this unit. In the next unit we shall continue our discussion on 'counting' from a slightly different perspective. Let us now summarise what we have covered in this unit.

3.6 SUMMARY

In this unit, you have studied the following points.

1. The pigeonhole principle, stated in several forms, its proof, and its applications.
2. The generalized pigeonhole principle, its proof, and applications.
3. The inclusion-exclusion principle, and its proof.
4. Finding the number of surjective functions, the discrete probability and the number of derangements, by using the inclusion-exclusion principle.

3.7 SOLUTIONS /ANSWERS

E1) By drawing lines parallel to the sides and through the points trisecting each side, we can divide the equilateral triangle into 9 equilateral triangles of side 1 cm. Thus, if 10 points are chosen, at least two of them must lie in one of the 9 triangles.

E2) 5 persons can be paired in C(5, 2) =10 ways. Hence, if pairs are invited 11 times, at least one pair must have been invited twice or more times, by the pigeonhole principle.

E3) Four persons can be arranged in a line in 4! = 24 ways. Hence, if we consider 25 occasions, at least on two occasions the same ordering in the queue must have been found, by the pigeonhole principle.

E4) Consider the following grouping of numbers:

{1, 2, 4, 8, 16}, {3, 9, 18}, {5, 15}, {6, 12}, {7, 14}, {10, 20}, {11}, {13}, {17}, {19}.

There are 10 groupings, exhausting all the 20 integers from 1 to 20. If 11 numbers are chosen it is impossible to select at most one from each group. So two numbers have to be selected from some group. Obviously one of them will divide the other.

E5) Suppose $x_1, x_2, \ldots, x_{15}$ are the number of balls in the 15 boxes, listed in increasing order, assuming that all these numbers are different. Then, clearly, $x_i \geq i-1$ for i = 1, 2,…,15. But then, $\sum_{i=1}^{15} x_i \geq 14.15/2 = 105$.

But the total number of balls is only 100, a contradiction. Thus, the x_is cannot all be different.

E6) In the sequence $a_1, a_2, \ldots, a_n$, there are (n+1)/2 odd numbers and (n–1)/2 even numbers because n is odd. Hence, it is impossible to pair all the a_is with numbers from 1, 2,…, n with opposite parity (evenness and oddness). Hence, in at least one pair (i, a_i), both the numbers will be of the same parity. This means that the factor $(a_i - i)$ will be even, and hence the product will be even.

E7) Consider the seven colours as containers, and the integers getting the respective colour as their contents. Then we have a distribution of an infinite number of objects among 7 containers. Hence, by Theorem 5, at least one container must have an infinity of objects, that is, the colour of that container must have been used an infinite number of times.

E8) Let H be the family of 5-element subsets B of A. For each B in H, let f(B) be the sum of the numbers in B. Obviously, we must have

$f(B) \geq 1 + 2 + 3 + 4 + 5 = 15$, and $f(B) \leq 46 + 47 + 48 + 49 + 50 = 240$.

Hence, $f: H \to T$ where T = {15, 16,…, 240}.

Since $|T| = 226$ and $|H| = C(10,5) = 252$, by Theorem 4, H contains different sets with the same image under f, that is different sets, the sums of whose elements are equal.

E9) The 100 collections can be considered as containers. There are an infinity of even numbers. When these even numbers are distributed into 100 containers, at least one container must have an infinity of them, by Theorem 5.

E10) The inclusion-exclusion formula can be rewritten as

$$\left|\bar{A}_1 \cap \ldots \cap \bar{A}_n\right| = N - (S_1 - S_2 + \ldots + (-1)^{n-1} S_n.$$

Also, we know that $\left|\bar{A}_1 \cap \ldots \cap \bar{A}_n\right| = N - |A_1 \cup \ldots \cup A_n|$.

Hence the result.

E11) Let the objects be the integers 0, 1,…, 999. Let A_1 be the set of numbers divisible by 5, and A_2 the set of numbers divisible by 7. Now, N = 1000, $|A_1|$ = 200, $|A_2|$ =143 and $|A_1 \cap A_2| = 29$. So, by Theorem 7, the answer is 1000 – 200 – 143 + 29 = 686.

E12) The answer to this problem is clearly the number of functions from an 8-element set (the set of people) onto a set of 4-elements (the set of floors). This number is

$$\sum_{i-0}^{4} C(4,i)(4-i)^8 = 4^8 - 4.3^8 + 6.2^8 - 4.1^8.$$

E13) We can choose three digits in C(10,3) = 120 ways.
The number of 6-digit numbers, using all the three digits, is the same as the number of functions from a 6-set onto a 3-set. This number is

$3^6–3.2^6 + 3.1^6 = 540$.

Hence, the answer is 120.540 = 64800. This will include numbers starting with 0 also.

E14) The total number of ways in which 13 cards can be chosen from a deck of 52 cards is C(52, 13).

If A_i is a choice of cards, none of which are from the ith suit, for i = 1,2,3,4, then $|A_i| = C(39,13), |A_i \cap A_j| = C(26,13)$, and $C(A_i \cap A_j \cap A_k) = C(13,13)$.

So, $\left|\cap \bar{A}_i\right| = C(52,13) - 4C(39,13) + C(4,2)C(26,13) - C(4,3)\ C(13, 13)$

Hence, the required probability is $\dfrac{\left|\cap \bar{A}_i\right|}{C(52,13)}$.

E15) If A,B,C,D are the integers divisible by 2, 3, 5, 7, respectively, then

$$\left|\bar{A} \cap ... \cap \bar{D}\right| = 10{,}000 - \left[\frac{10000}{2}\right] - \left[\frac{10000}{3}\right] - \left[\frac{10000}{5}\right] - \left[\frac{10000}{7}\right]$$

$$+ \left[\frac{10000}{6}\right] + \left[\frac{10000}{15}\right] + \left[\frac{10000}{35}\right] + \left[\frac{10000}{14}\right] + \left[\frac{10000}{21}\right] + \left[\frac{10000}{10}\right]$$

$$- \left[\frac{10000}{30}\right] - \left[\frac{10000}{42}\right] - \left[\frac{10000}{105}\right] - \left[\frac{10000}{70}\right] + \left[\frac{10000}{210}\right]$$

= 2285, where [x] denotes the greatest integer $\leq$ x.

Hence, the required probability is $\dfrac{2285}{10000} = 0.23$.

E16) If A_r is the event that the rth person gets the right coat, then by Theorem 7,

$$\left|\cap \bar{A}_i\right| = n! - \sum_r |A_r| + \sum_{r,s} |A_r \cap A_s| - ...$$

$$= n! – n(n–1)! + C(n,2)(n–2)! – C(n,3)(n–3)!+…$$

$$= C(n,2)(n-2)! - C(n,3)(n-3)! + \ldots$$

$$= n! \left(\sum_{r=0}^{n} (-1)^r \frac{1}{r!} \right)$$

The number of ways in which only one person receives the correct coat is the sum of all possible intersections of (n–1) $\bar{A}_i$ s. This is

$$n!\ n(n-1)! \left(\sum_{r=0}^{n-1} (-1)^r \frac{1}{r!} \right) = n! \left(\sum_{r=0}^{n-1} (-1)^r \frac{1}{r!} \right).$$

E17) 1, 3, 5, 7, 9 are the odd integers.
By Theorem 11(i), the required number of ways is
$9! - C(5, 1)8! + C(5, 2)7! - C(5,3)6! + C(5, 4)5! - C(5, 5)4!$

E18) By Theorem 11(ii), the required number of permutations is
$C(9,4)D_{9-4} = C(9,4)D_5$.

UNIT 4 PARTITIONS AND DISTRIBUTIONS

Structure **Page No.**

4.0 INTRODUCTION

In the last two units we have exposed you to a variety of combinatorial techniques. In this unit we look at a few more ways of counting arrangements of objects when order matters, and when it doesn't.

In Sec. 4.2, we focus on the ways in which a natural number can be written as a sum of natural numbers. In the process you will be introduced to a useful 'recurrence relation'.

We link this, in Sec. 4.3, with the different ways in which n objects can be distributed among m containers. As you will see, there are four broad possible kinds of distributions. In each case, we consider ways of counting all the distributions. In the process you will also be introduced to Stirling numbers.

With this unit we come to the end of our discussion on counting techniques. Some of the problems you have studied here will be looked at from different approaches in our later course MCS-033.

You should attempt the assignment based on the course after studying this unit, and this block.

4.1 OBJECTIVES

After going through this unit, you should be able to:

- define an integer partition, and count the number of partitions of an integer;
- count the number of ways of distributing distinguishable and indistinguishable objects, respectively, into distinguishable containers;
- count the number of ways of distributing distinguishable and indistinguishable objects, respectively, into indistinguishable containers.

4.2 INTEGER PARTITIONS

Suppose a detergent manufacturer wants to promote her product by giving a gift token with 100 bars out of the whole stock. The lucky persons among her customers will get the gift. Some of them may buy more than one bar at a time with the hope of getting gifts. In how many ways can the 100 gift tokens get distributed? One possible way is that all the 100 bars with gifts are bought by 100 different customers. We can indicate this situation by $100 = \underbrace{1+1+\ldots+1}_{100 \text{ times}}$. Another possibility is that somebody buys 2 bars, somebody else buys 3 bars, and the remaining 95 bars are distributed amongst 95 different people. We are not interested in the order in which the bars are bought. For example, here we are not interested in whether the person who bought 2 bars bought them before the person who bought the 3 bars. So, we can indicate this situation by $100 = \underbrace{1+1+...+1}_{95\text{ times}} + 2 + 3$. More generally, we can indicate each way of distributing the 100 bars with gifts by 100 = $p_1 + p_2 + p_2 + \ldots + p_k$, where the p_i are natural numbers, and $p_1 \le p_2 \le \ldots \le p_k$. Each way of writing 100 in this form is called an **integer partition** of 100. More generally, we have the following definition.

Definition: Any representation of $n \in \mathbf{N}$ as a sum of positive integers in non-increasing order is called a **partition** (or **integer partition**) of n. Each such partition can be written in the form $n = p_1 + p_2 + \ldots + p_k$, where $p_1 \le p_2 \le \ldots \le p_k$.

Here, $p_1, p_2, \ldots, p_k$ are called the **parts** of the partition, and the **number of parts** of the partition is k.

While we chose 100 in the example above, it is really a huge number in the context of integer partitions. Let us consider a smaller number, say 5. How many partitions of 5 can you think of? There are 7 altogether, namely, 5, 1+4, 2+3, 1+1+3, 1+2+2, 1+1+1+2 and 1+1+1+1+1.

In books, you will often come across the notation p(n) for the number of partitions of n.

If **we represent the number of partitions of the integer n by P_n,** we have shown that $P_5 = 7$. These partitions have 1, 2, 2, 3, 3, 4 and 5 parts, respectively.

If **we represent the number of partitions of n with exactly k parts by** P_n^k, then we have $P_5^1 = 1, P_5^2 = 2, P_5^3 = 2, P_5^4 = 1, P_5^5 = 1$.

To check your understanding of the material so far, try the following exercises.

E1) Write down all the partitions of 7. Also find P_7^4 and P_7^5.

E2) Let us consider the situation of the detergent manufacturer again. Suppose she only wants to distribute 10 gift tokens in 5 specific sales districts, where the sales are low. What is the number of ways of doing this?

You may wonder if you've found all the partitions in E2. One way to check is by finding out the required number in terms of partitions of smaller numbers, which may be easier to find. One such relation between partitions of n and n+1, n+2, etc. is given in the following theorem.

Theorem 1: $P_n^1 + P_n^2 + ... + P_n^k = P_{n+k}^k, P_n^1 = P_n^n = 1$, for $1 \le k \le n$, that is, the number of **partitions of n with at most k parts** is the same as the number of **partitions of n+k with exactly k parts**.

Before we begin the proof of this theorem, let us consider an example. Let us take n = 4, k = 3. According to Theorem 1, we must have $P_4^1 + P_4^2 + P_4^3 = P_7^3$. Note that $P_4^1 + P_4^2 + P_4^3$ is the total number of partitions of 4 with 1, 2 or 3 parts, i.e., the number of partitions with **at most** 3 parts. There is one partition of 4 with one part, 4 = 4. Let us write this as a 3-tuple, (4, 0, 0), adding two more zeroes since we are considering partitions with at most **3** parts. If we add 1 to all the entries of this 3-tuple, we get (4+1,0+1,0+1) = (5,1,1) and (1+1+5 is a partition of 7 with three parts. Similarly, consider the partition 4 = 1+3 of 4 into two parts. Again, we can write this as (1, 3, 0).

Now, if we add 1 to each of the entries, we get (2, 4, 1) and 1+2+4 is a partition of 7 into three parts. Conversely, if we take the partition 7 = 1 + 3 +3 of 7 into three parts, write it as (1, 3, 3) and **subtract** 1 from all the entries, we get (0, 2, 2) which corresponds to the partition 4 = 2+2 of 4 into 2 parts. In this way, we can match every partition of 4 with **at most 3** parts with a partition of 7 with **exactly 3** parts, and vice versa. This is the basic idea behind our proof of Theorem 1, which we now give.

Proof of Theorem 1: The cases $P_n^1 = 1 = P_n^n$ follow from the definition.

We will prove the general formula now. Let M be the set of partitions of n having k or less parts. We can consider each partition belonging to M as a k-tuple after adding as many zeroes as necessary. Define the mapping

$$(p_1, p_2,\ldots, p_m, \underbrace{0,0,\ldots,0}_{\substack{(k-m)\\ \text{times}}}) \mapsto (p_1+1, p_2+1,\ldots,p_m + 1, \underbrace{1,1,\ldots,1}_{\substack{(k-m)\\ \text{times}}}), m \le k$$

from M into the set M′ of partitions of n+k into exactly k parts. This mapping is bijective, since

i) two distinct k-tuples in M are mapped onto two distinct k-tuples in M′;

ii) every k-tuple in M′ is the image of a k-tuple of M. This is because, if

$(p_1, p_2,\ldots,p_k)$ is a partition of n+k with k parts, then it is the image of

$(p_1 - 1, p_2 - 1,\ldots, p_k - 1)$ under the mapping above.

Therefore, $|M| = P_n^1 + \ldots + P_n^k = |M'| = P_{n+k}^k$, and the theorem is proved.

Note that $P_n^k = 0$ **if n < k**, since there is no partition of n with k parts if n < k. Also, $\mathbf{P_n^n = P_n^1 = 1}$.

The formula in Theorem 1 is an identity which allows us to find P_n^r from values of P_m^k, where m < n, k ≤ r. This is why it is also called a **recurrence relation** for P_n^k.

Theorem 1 is every useful. For instance, to verify your count in E2, you can use it because $P_{10}^5 = P_5^1 + P_5^2 + \ldots + P_5^5 = 7$.

From Theorem 1, the P_n^ks may be calculated recursively as shown in Table 1.

Table 1 : P_n^k for 1 ≤ n, k ≤ 6

n \ k	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	0	0	0
4	1	2	1	1	0	0
5	1	2	2	1	1	0
6	1	3	3	2	1	1

In this table, the second entry in the row corresponding to n = 4 is P_4^2. By Theorem 1, $P_4^2 = P_2^1 + P_2^2$, which is the sum of the entries in the row corresponding to n = 2.

Similarly, P_6^3 is the sum of the entries in the row corresponding to n = 3.

Now, here is an exercise for you.

E3) Use Table 1 to find the values of $P_7^k, 1 \le k \le 6$.

The partition of a number n into k parts also tells us how n objects can be distributed among k boxes. We will now consider all possibilities of such distributions.

4.3 DISTRIBUTIONS

By a distribution we mean a way of placing several objects into a number of containers. For example, consider the distribution of 6 balls among 3 boxes. We may have all 6 balls of different shapes, sizes and colours, i.e., they are all distinguishable. Or, all the balls could be exactly the same, i.e., they are all indistinguishable.

Similarly, all 3 boxes may look different, or all 3 could be exactly the same. So, we see that there are 4 possibilities here.

In fact, we have the following possibilities for any set of n objects and m boxes.

Case 1: The objects are distinguishable, and so are the boxes;

Case 2: The objects are distinguishable and the boxes are indistinguishable;

Case 3: The objects are indistinguishable and the boxes are distinguishable;

Case 4: The objects are indistinguishable, and so are the boxes.

You may be surprised to know that in each of the cases the number of such distributions is different. In fact, the distribution problem is to count all possible distributions in any of these situations, or in a combination of these cases.

A general guideline for modelling a 'distribution problem' is that a distribution of distinct objects corresponds to an arrangement, and a distribution of identical objects corresponds to a selection. Let us consider examples of each of the four cases given above.

(a) There are twenty students and four colleges. In how many ways can the students be accommodated in the four colleges?

In this example the students, as well as the colleges, are clearly distinguishable. This comes under Case (1).

(b) Suppose we want to group 100 students into 10 groups of 10 each for the purpose of a medical examination. In how many ways can this be done?

Here the groups are indistinguishable, though the students in them are distinguishable. Hence, this falls under Case (2).

(c) An employer wants to distribute 100 one-rupee notes among 6 employees. What is the number of ways of doing this?

Though the one-rupee notes can be distinguished by their distinct numbers, we don't consider them to be distinguishable as far as their use is concerned. The employees, of course, are distinguishable. Hence, this is an example of Case (3).

(d) There are 1000 one-rupee notes. In how many ways can they be bundled into 20 bundles?

As before, the rupee notes are treated as indistinguishable. Clearly, the bundles are, by themselves, not distinguishable. Only the quantity in each may vary. Hence, this falls under Case (4).

Let us consider each case in some detail now.

4.3.1 Distinguishable Objects into Distinguishable Containers

Let us consider the example (A) above. Since any number of students can be put in a college, there are $20 \times 20 \times 20 \times 20$ possibilities, by the multiplication principle.

More generally, suppose we are distributing n objects into m containers, both being distinguishable. Then the total number of such distributions is n^m.

Let us look at an example.

Example 1: Show that the number of words of length n on an alphabet of m letters is m^n.

Solution: The m letters of the alphabet can be used any number of times in a word of n letters. The word can be considered as n ordered boxes, each holding a letter from the alphabet. The boxes become distinguishable because they are ‘ordered’. The letters of the alphabet are clearly distinguishable. So, the number of ways of doing this is m^n.

* * *

Several people are confused while solving the problem above. They tend to take the m letters as the containers instead! Let’s consider another example.

Example 2: Suppose we have a set S with n objects. An m-sample from this set S is an ordered arrangement of m letters taken from S, with replacement at every draw, in m draws. Find the number of m-samples from an n-element set.

Solution: Every m-sample is a word of length m from the ‘alphabet’ S containing n letters. Hence, the required number is n^m.

* * *

Now here are some exercises for you to solve.

E4) Find the number of three-letter words that can be formed using the letters of the English alphabet. How many of them end in x? How many of them have a vowel in the middle position?

E5) How many five-digit numbers are even? How many five-digit numbers are composed of only odd digits?

E6) There are 4 women and 5 men. A committee of three, a president, a vice-president, and a secretary, has to be formed from them. In how many ways can this be done if

i) the vice-president should be a woman?

ii) exactly one out of the vice-president and the secretary should be a woman?

iii) there is at least one woman in the committee?

Now suppose, we want to find the number of distributions of n distinguishable objects into m distinguishable containers, with **the extra condition** that no container should

contain more than one object. It is clear that this requires $m \geq n$. Then we can get all these arrangements by first choosing n containers to contain exactly one object, and then permuting the n objects among the chosen containers. This can be done in C(m, n). n! = P(m, n) ways.
So, we have proved the following result.

Theorem 2: The number of ways of distributing n distinguishable objects into m distinguishable containers such that no container contains more than one object is P(m, n).

For example, the cardinality of the set of 5-digit numbers with all digits being distinct odd numbers is P (5,5). This is because the possible digits are 1, 3, 5, 7, 9.

Why don't you try an exercise now?

E7) Find the number of m-letter words with distinct letters, all taken from an alphabet with n letters, where $n \geq m$. Is this different from the number of injective mappings from an m-element set into an n-element set, where $n \geq m$? Give reasons for your answer.

Let us now consider the second type of distribution.

4.3.2 Distinguishable Objects into Indistinguishable Containers

Here we shall find the number of ways of distributing n distinguishable objects into m indistinguishable containers. For this, we first find the number when exactly k of the containers are occupied. This brings us to Stirling numbers of the second kind, named after James Stirling (1692-1770).

Suppose $n \geq m$. The number of distributions of n distinguishable objects into m indistinguishable containers **such that no container is empty** is represented by $\mathbf{S_n^m}$. This number is called the Stirling number of the second kind. As you can see, this is also the number of partitions of a set of n objects into m classes.

Definition: For natural numbers n and m, the **Stirling number of the second kind**, S_n^m, is the number of partitions of an n-element set into exactly m parts.

Note that:

i) $S_n^m = 0$ if $n < m$, for, if the number of containers exceeds the number of objects, then it is impossible to have all the containers non-empty.

ii) $S_n^n = 1$, since there is only one way of putting n distinguishable objects in n indistinguishable boxes so that no box is empty.

iii) $S_n^1 = 1$.

Now, we shall use the inclusion exclusion principle to find the value of S_n^m.

Theorem 3: $S_n^m = \dfrac{1}{m!}\sum\limits_{k=0}^{m}(-1)^k C(m, m-k)(m-k)^n$, $n \geq m$.

Proof: If the m classes are distinguishable, the number of partitions is the same as the number of functions from an n-element set onto an m-element set. As the classes are distinguishable here, we have to divide this number by m!. The result follows from Theorem 8, Unit 3.

For example, to obtain the Stirling number, S_5^3, we know that the number of functions from a 5-element set onto a three-element set is 150. So, by Theorem 3, $S_5^3 = 150/3!$ = 25.

Remark: You may be wondering how we have jumped straightaway to the Stirling numbers of the second kind. What about the first kind? We won't be using them in any way here. However, for a the sake of completeness, we define **Stirling numbers of the first kind, s(n, k),** as follows.

For a positive integer n, and $0 \le k \le n$, s(n, k) is the coefficient of x^k in the expansion of the multinomial $x(x-1)(x-2)\ldots(x-n+1)$.

Getting back to S_n^m, you may feel that the formula in Theorem 3 is a little cumbersome. Sometimes, the following recurrence relation for S_n^m may be more useful.

Theorem 4: If $1 < m \le n$, then $S_{n+1}^m = S_n^{m-1} + mS_n^m$.

Proof: Let us take n+1 objects, mark one of them, and consider the distribution of these n+1 objects into m indistinguishable containers. Then we have 2 situations.

Case (1) (The marked object is placed in one container without any other objects.): In this case, the remaining n objects can be placed in (m – 1) containers in S_n^{m-1} ways.

Case (2) (The marked object is placed with at least one more object in a container.): In this case, we can first distribute the n unmarked objects into m containers, and then put the marked objects m to one of these m containers. So, the number of such partitions is $m\,S_n^m$.

Therefore, by the addition principle, we get $S_{n+1}^m = S_n^{m-1} + mS_n^m$.

There is a generalisation of Theorem 4 that is of independent interest, which we now state.

Theorem 5: $S_{n+1}^m = \sum_{k=0}^{n} C(n,k).S_k^{m-1}$

Proof: Let us mark one object in a set of (n+1) objects. Suppose the marked object is present in a box with (n – k + 1) elements, where $m - 1 \le k \le n$. Then we can choose n – k more objects to go with the marked object in C(n, n – k) ways. The remaining k objects can be distributed into $(m-1)^n$ boxes in S_k^{m-1} ways. So the number of ways of distributing the n – k objects is C(n, n – k) S_k^{m-1}. The result now follows from the addition principle by allowing k to vary from 0 to n.

Let us see some examples of the use of these recurrences.

Example 3: Calculate S_3^2 and S_4^2.

Solution: Using Theorem 4, we get $S_3^2 = S_2^1 + 2 \times S_2^2 = 1 + 2 \times 1 = 3$, and $S_4^2 = S_3^1 + 2S_3^2 = 1 + 2 \times 3 = 7$.

* * *

Now let us find what we had started with in this sub-section.

Theorem 6: The number of ways of distributing n distinguishable objects into m indistinguishable containers is $S_n^1 + S_n^2 + \ldots + S_n^m$, where $n \geq m$. (Note that here we do not insist that no container is empty.)

Proof: When we distribute n distinguishable objects into m indistinguishable containers there are m cases. Case (k) is that exactly k containers are non-empty. Here k varies from 1 to m. The number of distributions in Case (k) is S_n^k. The result now follows from the addition principle.

Let us consider an example.

Example 4: In how many ways can 20 students be grouped into 3 groups?

Solution: Theorem 6 says that this can be done in $S_{20}^1 + S_{20}^2 + S_{20}^3$ ways.

Now, using Theorem 3, we get this number to be

$$1 + \frac{1}{2}\sum_{k=0}^{2}(-1)^k C(2,2-k)(2-k)^{20} + \frac{1}{6}\sum_{k=0}^{3}(-1)^k C(3,3-k)(3-k)^{20}$$

= 581,130,734.

* * *

Try some exercises now.

E8) Find the number of surjective functions from an n-element set onto an m-element set.

E9) Find the number of ways of placing n people in n – 1 rooms, no room being empty.

Let us now consider the third possibility for distributing objects into containers.

4.3.3 Indistinguishable Objects into Distinguishable Containers

Suppose there are n indistinguishable objects and m distinguishable containers. As the objects are indistinguishable, the distributions depend only on the number of objects in each container. As the containers are distinguishable, they can be assumed to be arranged in a line. Hence, the number of distributions is the number of ways of writing the number n as the sum $x_1+x_2+\ldots+x_m$, where the x_i's are non-negative integers.

We have covered this situation in Theorem 5 of Unit 2. Over there we have shown that **the number of distributions of n indistinguishable objects into m distinguishable containers is C (m+n–1, n)**. In particular, the number of non-negative integral solutions of the equation $x_1+x_2+\ldots+x_m = n$ is C (m+n–1, n).

Incidentally, we note that the number of distributions of n indistinguishable objects into m distinguishable containers **with at most one object per container** is C (m, n).

Let us consider an example.

Example 5: How many distinct solutions are there of x + y + z + w = 10

i) in non-negative integers?

ii) in positive integers?

Solution:

i) From the result quoted above, the answer is C(4 + 10 – 1, 10) = 286.

ii) We want x, y, z, w to be positive. Hence, we can write them respectively as X+1, Y+1, Z+1, W+1, where X, Y, Z, W are non-negative. Hence we want the number of non-negative solutions of the equation X+1+Y+1+Z+1+W+1=10, i.e., X+Y+Z+W= 6. The answer, now, is C (4 + 6 – 1, 6) = 84.
Try some exercises now.

E10) Show that the number of positive solutions of the equation $x_1+x_2+\ldots+x_n = m$ is C(m – 1, m – n).

E11) In how many ways can an employer distribute 100 one-rupee notes among 6 employees so that each gets at least one note?

Let us now consider the fourth case.

4.3.4 Indistinguishable Objects into Indistinguishable Containers

Suppose there are n indistinguishable objects and m indistinguishable containers. Any distribution is determined purely by an **unordered** m-tuple of non-negative integers with sum n. This is equivalent to the number of increasing sequences of length m of non-negative integers with sum n. But this is precisely the number of partitions of the integer n with at most m parts, viz., $P_n^1 + P_n^2 + \ldots + P_n^m = P_{n+m}^m$, from Theorem 1 of this unit.

Let us consider an example of this case.

Example 6: In how many ways can 20 identical books be placed in 4 identical boxes?

Solution: The answer is $P_{20}^1 + P_{20}^2 + P_{20}^3 + P_{20}^4 = P_{24}^4$

Why don't you try some exercises now?

E12) In how many ways can 1000 one-rupee notes be bundled into a maximum of 20 bundles?

E13) A car manufacture has 5 service centres in a city. 10 identical cars were served in these centres for a particular mechanical defect. In how many ways could the cars have been distributed at the various centres?

With this we have come to the end of this unit. Let us take a quick look at what we have studied in this unit.

4.4 SUMMARY

1. A partition of $n \in \mathbf{N}$ into k parts is $x_1+x_2+\ldots+x_k = n$, where $x_1 \le x_2 \le \ldots \le x_k$. P_n is the set of all partitions of n, and P_n^k is the set of all partitions into exactly k parts.

2. The proof and applications of the recurrence relation, $P_n^1 + P_n^2 + \ldots + P_n^k = P_{n+k}^k, P_n^1 = P_n^n = 1, 1 \le k \le n.$

3. The number of ways of distributing n objects into m containers is :

i) n^m, if the objects and containers are distinguishable.

ii) $\sum_{i=1}^{m} S_n^i$, if the objects are distinguishable but the containers are not.

(Here S_j^i is a Stirling number of the second kind).

iii) C(m+n–1, n) if the objects are not distinguishable but the containers are distinguishable.

iv) P_{n+m}^m, if neither the objects nor the containers are distinguishable.

Further, in (i) above, if there is an extra requirement that each container contain at most one object, then the number of distributions is P(m, n). Again, in (iii) above, with the same extra requirement, the number of distributions is C(m, n).

4.5 SOLUTIONS /ANSWERS

E1) In the table below we give all possible partitions of 7.

Table 2

Number of parts	Partitions
1	7
2	1+6, 2+5, 3+4
3	1+1+5, 1+2+4, 1+3+3, 2+2+3
4	1+1+1+4, 1+1+2+3, 1+2+2+2
5	1+1+1+1+3, 1+1+1+2+2
6	1+1+1+1+1+2
7	1+1+1+1+1+1+1

From the table, we see that $P_7^4 = 3,\ P_7^5 = 2.$

E2) The required number is $P_{10}^5 = 7.$

E3) $P_7^1 = 1 = P_7^7.$

$P_7^2 = P_5^1 + P_5^2 = 1 + 2 = 3$, from Table 1.

$P_7^3 = P_4^1 + P_4^2 + P_4^3 = 1 + 2 + 1 = 4$, from Table 1.

Similarly, $P_7^4 = P_3^1 + P_3^2 + P_3^3 + P_3^4 = 3,\ P_7^5 = P_2^1 + P_2^2 = 2$ and $P_7^6 = P_1^1 = 1.$

E4) The 26 letters are distinguishable objects. We have to fill then in three distinguishable containers, viz., the first, second, and third positions of a three-lettered word. The solution is 26^3.

If the last letter is to be x, the number is only $26^2 \times 1$.

If the middle letter is a vowel, then by the multiplication principle, the answer is $26 \times 5 \times 26$.

E5) The total number of even numbers is $9 \times 10 \times 10 \times 10 \times 5 = 45{,}000$, since the last digit can only by 0, 2, 4, 6 or 8.

The number of 5-digit numbers composed of only odd digits (i.e., 1, 3, 5, 7, 9) is clearly $5^5 = 3125$.

E6) i) We can choose a woman for vice-president in 4 ways. To fill the remaining 2 positions we can select 2 from the remaining 8 persons in $8 \times 7 = 56$ ways. Hence, the required number is $4 \times 56 = 224$.

ii) If the vice-president is a woman (chosen in 4 ways), others can be selected in $5 \times 4 = 20$ ways. Similarly, if the woman is a secretary, the others can be chosen in 20 ways. Hence, by the addition and multiplication principles, the answer is $20 \times 4 + 20 \times 4 = 160$.

iii) Without any restriction, three can be selected in $9 \times 8 \times 7 = 504$ ways. If no woman is to be selected, then it can be done in $5 \times 4 \times 3 = 60$ ways. What we need is the complement of this. Thus, the required answer is $504 - 60 = 444$.

E7) If the alphabet has n letters, the m-letter words with distinct letters can be formed in $n(n-1)(n-2)\ldots(n-m+1) = P(n, m)$ ways.

Now, in an injective mapping, images of distinct elements should be distinct (see Unit 1). There are n possible images for the first element of the m-set, n–1 possible images for the second, and so on. Hence, the number of such mappings is also P(n, m).

E8) Suppose $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$. If f is an onto function from N to M, then the inverse images, $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)$ constitute a partition of N into m classes. The number of ways in which this can be done is S_n^m, where the order of partition is immaterial. But, in functions, the order cannot be ignored. So, the distribution can be done in $m!.\,S_n^m$ ways.

E9) This is S_n^{n-1}. This can be done by putting one person each in n – 2 rooms and 2 persons in 1 room. This can be done in C(n, 2) ways. So $S_n^{n-1} = C(n,2)$.

E10) If a positive solution is $x_1, x_2, \ldots, x_n$, then it can be written as $X_1+1, X_2+1, \ldots, X_n + 1$, where the X_i's are non-negative. Thus, the required number is the number of non-negative solutions of $X_1+X_2+\ldots+X_n+n = m$, which is $C(n + m - n - 1, m - n) = C(m - 1, m - n)$.

E11) This is the number of positive solutions of $x_1+\ldots+x_6 = 100$.
So, the required number is $C(100 - 1, 100 - 6) = C(99, 94) = 71{,}523{,}144$.

E12) $P_{1000}^1 + P_{1000}^{20} + P_{1020}^{20}$.

Had the requirement been that there be exactly 20 bundles, then the number would have been P_{1000}^{20}.

E13) $P_{10}^1 + P_{10}^2 + P_{10}^3 + P_{10}^4 + P_{10}^5 = P_{15}^5$.

SOCIAS-IGNOU/P.O.10.5T/MAY, 2004